Proof by Induction

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Lets say we wanted to prove a fact about an arbitrary positive integer, like for any $n \geq 4$ that out of all the subsets of $\{1, 2, 3, \ldots, n\}$ a quarter of them have a sum which is a multiple of 4, a quarter have a sum which is one more than a multiple of 4, a quarter have a sum which is two more than a multiple of 4 and a quarter of them have a sum which is three more than a multiple of 4.

Since n can be any positive integer greater than n we need to be able to come up with a proof of this for any n, we will call this proof $P(n)$, you might notice that $\{1, \ldots n\}$ and $\{1, \ldots, n, n+1\}$ have a lot of subsets in common, so in the process of coming up with the proof $P(n)$ we probably know a lot about the subsets of $\{1, \ldots n+1\}$ and so instead of checking all those those common subsets again, we can just what we have found out in $P(n)$ to come up with the next proof $P(n+1)$.

The idea of induction is that we first prove the first case, which is usually called the **base case** and we are able to show that if we had come up with a proof $P(n)$ we can then come up with a proof $P(n+1)$ this is called the **inductive hypothesis** because we are hypothesising that we do have a proof $P(n)$ then from $P(0)$ we can prove $P(1)$ and from $P(1)$ we can prove $P(2)$.

$$P(0) \to P(1) \to P(2) \to P(3) \to \cdots \to P(n)$$

And so we can we prove our fact for any n.

With induction we can get away with only really working out the easiest cases (Which you should always be trying), for example that after trying all 24 subsets of $\{1, 2, 3, 4\}$ and summing them it turns out exactly 6 of them are a multiple of 4, 6 of them are one more than a multiple of 6, 6 of them are two more than a multiple of 4 and 6 or them are three more than a multiple of 4.

Now we can assume that our fact is true of $\{1, 2, 3, \ldots n\}$, please convince yourself that we only have to consider the new subsets of $\{1, 2, 3 \ldots, n, n+1\}$ which contain $n+1$, the rest of this proof is left as an exercise to the reader.

Note on writing up proofs: please be very clean when you are proving a fact using induction, here's an outline of what an induction should look like

1. "We are going to prove blah blah blah using induction"
2. **Base Case:** "we will check that the thing we are trying to prove is actually true for $n = 0$
3. **Inductive Hypothesis:** "assume that our claim is indeed true when $n = k$, now we will show that it must also be true for $n = k + 1$"
4. "by induction our our claim is true"

Problems

1. Prove that for any positive integer n

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

2. Prove that for any positive integer n

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

3. Prove that for any positive integer n

$$(1 + 2 + 3 + \cdots + n)^2 = (1^3 + 2^3 + 3^3 + \cdots + n^3)$$

   fun fact: $2025 = (1 + \cdots + 9)^2 = 1^3 + \cdots + 9^3$

4. For $n \geq 6$ show that you can divide one square up into n square pieces.

5. A circle is divided into regions by a number of chords. Prove that it is possible to color each region either black or white such that no two adjacent regions are the same color.

6. A circular island is divided into states by a number of chords of the circle. Consider a tour that starts and ends in the same state and ends in the same state without passing through the intersection of two borders. Prove that we must have passes an even number of borders.

7. Consider a $2^n \times 2^n$ chessboard with a corner removed. Prove that you can tile the whole chessboard with L-triminos (shown below).