IMO 2021 Solutions

Rick Han's Final Year

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Remark: I was just trying a A7, obviously I didn't solve it but I learn this:

When you have a wacky condition, it's like a puzzle hunt. Figure out what it's implying. Extraction, Extraction, Extraction!

Problem 1: Let $n \geq 100$ be an integer. Ivan writes the numbers $n, n+1, ..., 2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

We set up the following system of equations with a k to be determined later.

$$\begin{aligned} a+b &= (2k-1)^2 \\ b+c &= (2k)^2 \\ c+a &= (2k+1)^2 \end{aligned}$$

Solving this yields that $a = 2k^2 - 4k$, $b = 2k^2 + 1$, $c = 2k^2 + 4k$. Now if these exists k such that a, b, c are between n and $2n$ then there exists three cards which all pairwise sum to a perfect square and thus it is impossible for all three to be in different piles.

The important inequalities are (where k is an integer):

$$n \leq 2k^2 - 4k \text{ and } 2n \leq 2k^2 + 4k$$

We need to show that whenever $n \geq 100$ there is a suitable k for which this is true.

To do this we just show that the intervals

$$k^2 + 2k \leq n \leq 2k^2 - 4k, \quad k \geq 9$$

are overlapping, this is because $(k+1)^2 + 2(k+1) \leq 2k^2 - 4k$, whenever $k \geq 9$.

Problem 2: Show that the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}$$

The inequality is true when $n = 0$ and also true when $n = 1$. Keep this in mind.

If one of the terms is 0 then we remove it and it is equivalent to a smaller with $n - 1$ numbers.

Notice that for the left hand side only the relative differences between the terms is important.

Now visualize each of $x_i + x_j$ as a point on the number line, there will be n^2 of these. We're worried about the distance of all points from 0. This is minimized when 0 is the median of the set of points. So we actually only need to prove this case, as it would imply all other shifted cases.

If there is exactly one median, this implies that either 0 is a element, can't be the case, or there are two elements which are the negation of one another.

We can not safely remove them, and the direction of the inequality will stay the same.

This allows us to induct down by 2. Lovely

Remark: On a line, the point which minimizes the sum of the distances between itself and all other points is the median when $d = |a - b|$

The point is the average when $d = (a - b)^2$.

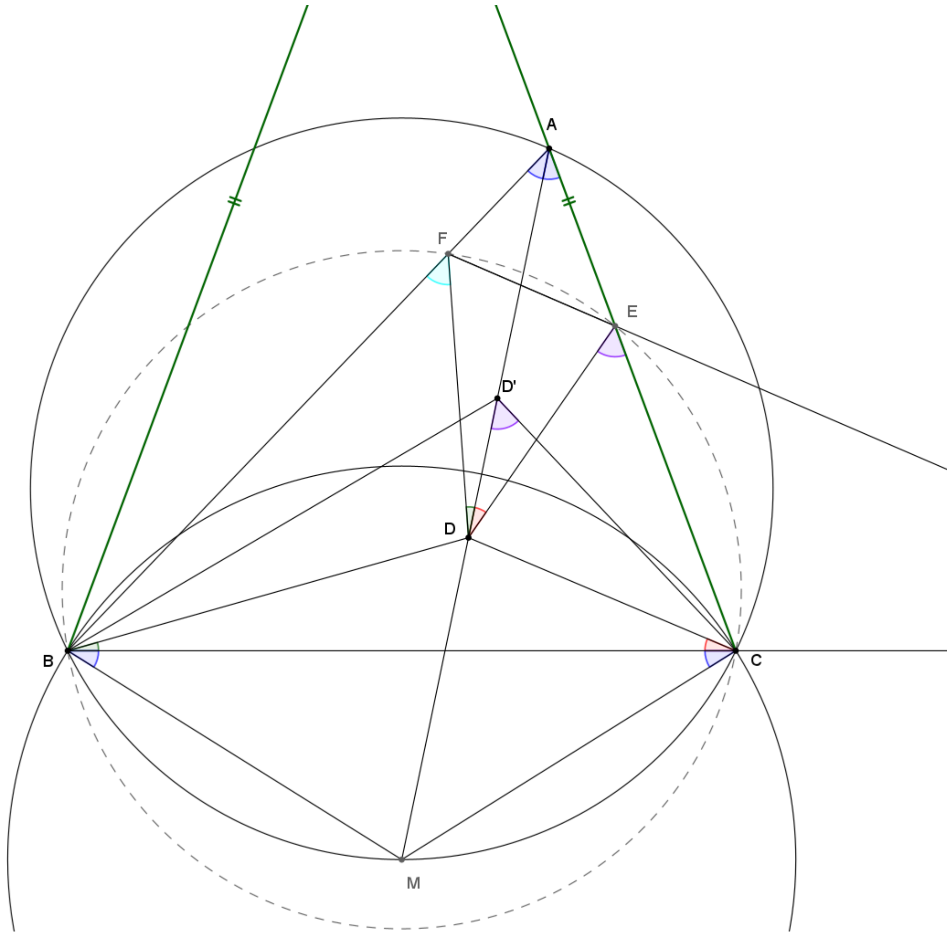
Learnt this one from competitive programming hehe.

Problem 3: Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of triangles ADC and EXD, respectively. Prove that the lines $BC, EF, \text{and}, O_1, O_2$ are concurrent.

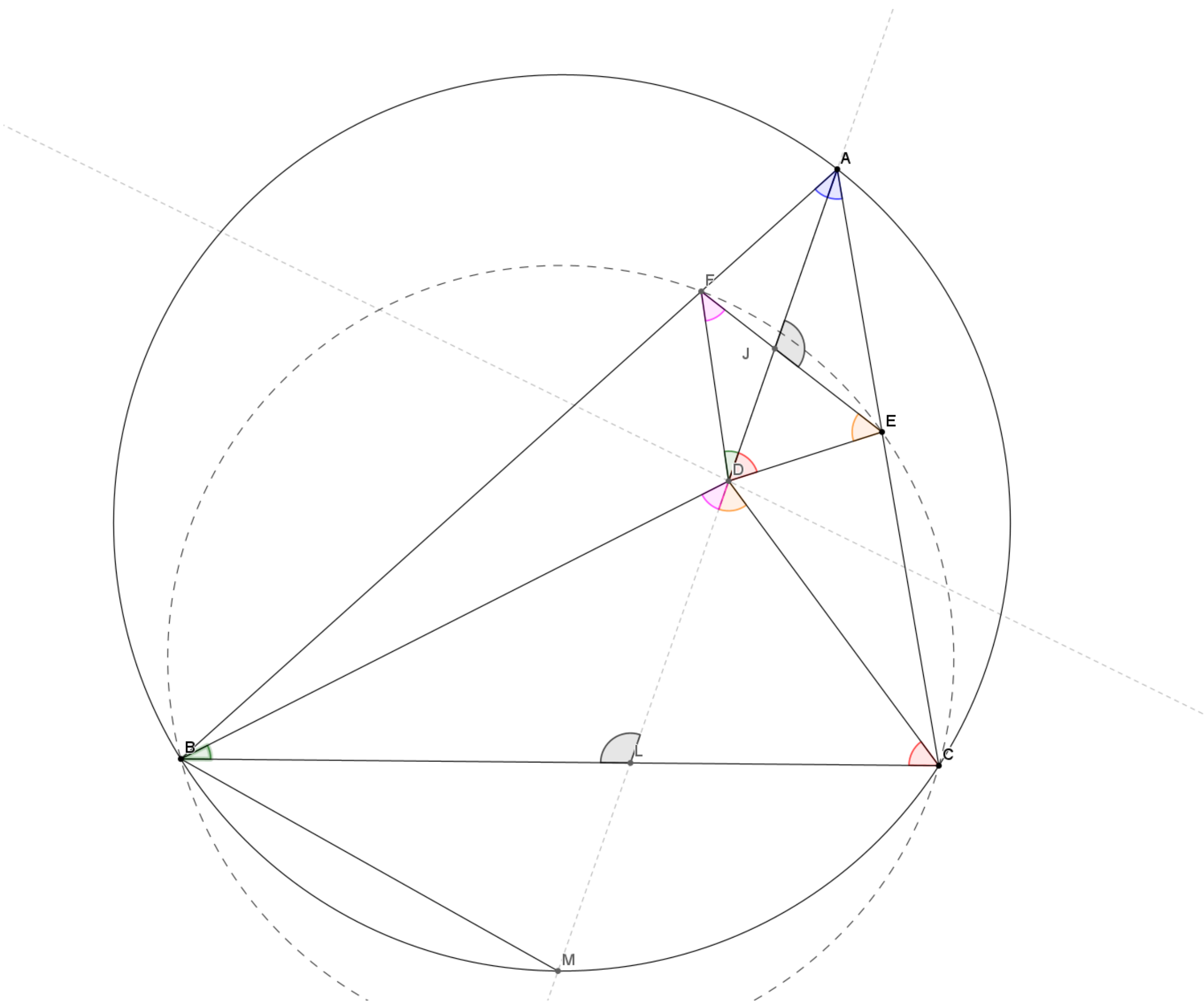
Claim 1: $FECB$ is cyclic.

Let M be the intersection of the angle bisector of $\angle BAC$ with (ABC) we have that M is the arc midpoint of $\overparen{BC}$. Invert D to D' about the circle centered at M with radius $MB = MC$. It follows that the purple angles are equal (red + blue) = purple.

It then must be that $(D'DCE)$ and $(DD'FB)$ are cyclic and so by power of a point $(FECB)$ is also cyclic.



It is clear from this that D and D' are isogonal conjugates. Now Let P be the intersection of $\overline{BC}$ and $\overline{FE}$



Claim 2: $DP^2 = PF \cdot PE = PC \cdot PB$

The equivalence of the red and blue angles is known. By symmetry the two black angles are equal, then the equivalence of the orange angles is forced. By similar reasoning the equivalence of the pink angles is forced.

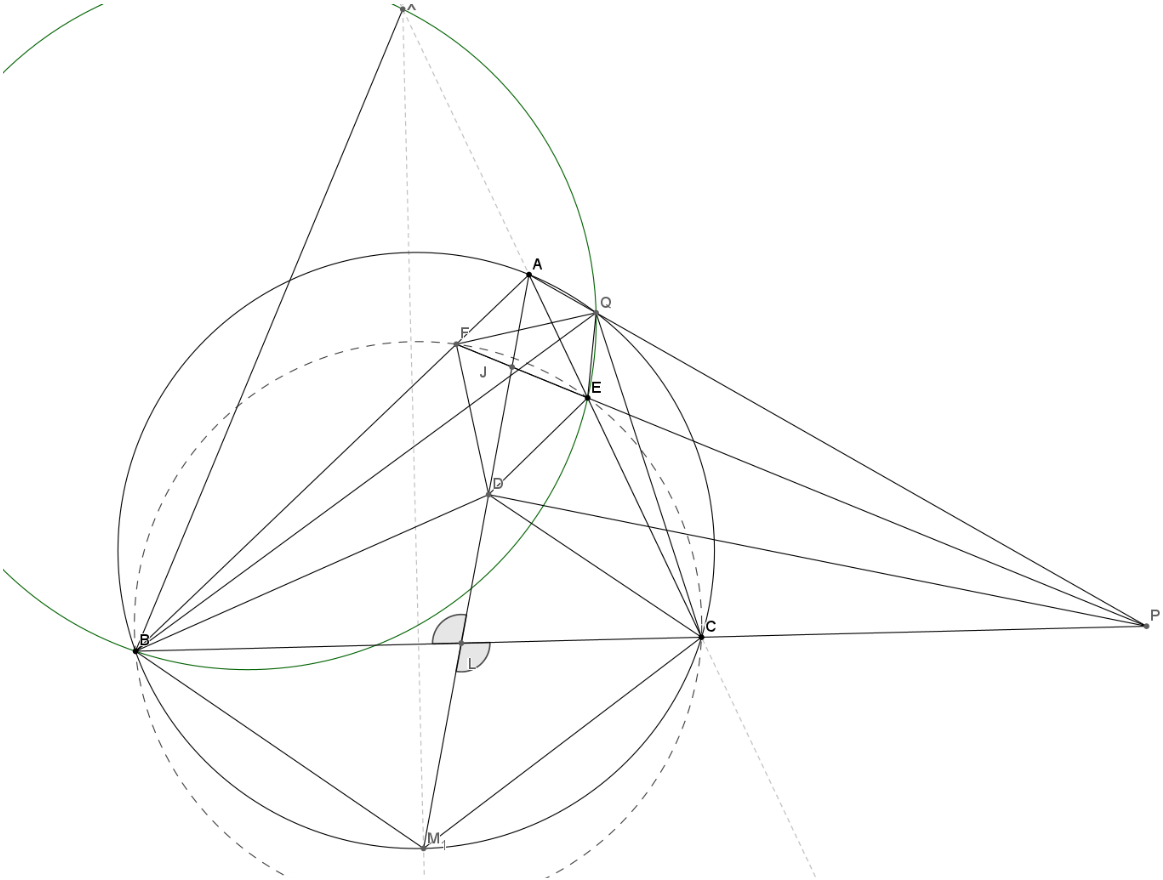
Then we have that

$$\angle CDE = 180° - \text{red} - \text{orange} = \text{green} + \text{pink}$$

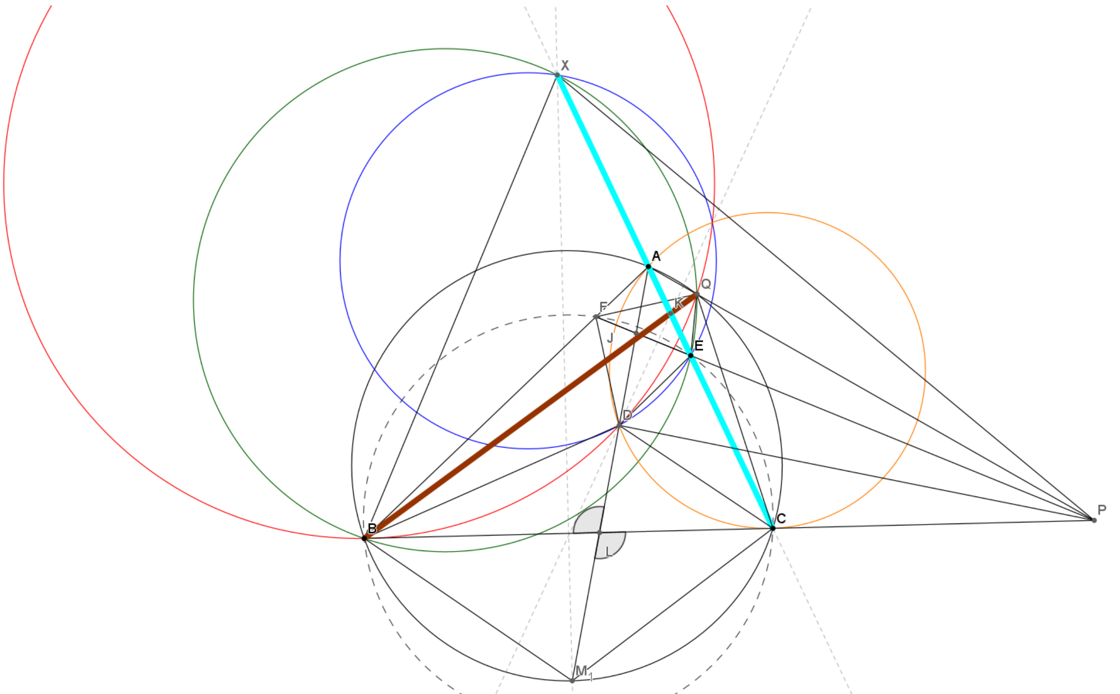
which is a sufficient condition for tangency. So (FED) and (BDC) are tangent at D, then by radical axis we have that FE and BC meet at P where DP is the common tangent of both circles. This implies claim 2.

Claim 3: Note also that those two black angles imply that $PJ = PL$

Now let's actually deal with the isos triangle. **Claim 4:** $BXQE$ is cyclic, this is by adding in angles A, B C and chasing around the diagram.



Claim 4: Center of Red and Orange circles are collinear with P, this is by inversion.



- Red, Black and Blue share radical Axis of Brown.
- Cyan is radical axis of Blue and Green.
- Cyan is radical axis of Black and Orange.

DK should then be the radical axis of Blue and Red this is by the Blue, Red and Green Circles.

DK should be the radical axis of Orange and Red. This is by the Red, Orange and Black circles.

So the centers of the circles Blue, Red and Orange are collinear, but we already know Red and Orange are collinear with P, so Blue and Orange are collinear.

Oh my gosh, I just solved 45 MOHS P3 Geo.

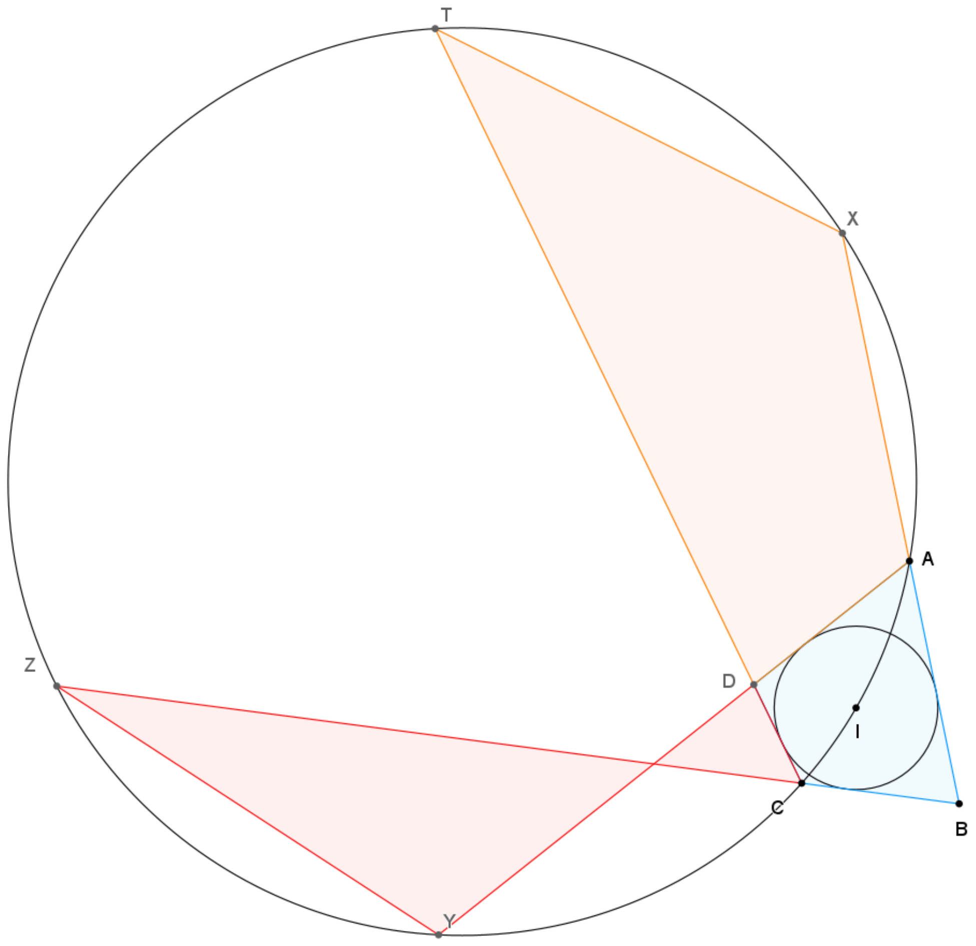
Remark: I only looked at Evan Chen's solution to get that $(AEFB)$ should be cyclic.

This should be motivated tho, by the desire to use the given condition in a nice way. Extraction! Extraction! Extraction!.

Problem 4: Let Γ be a circle with center I, and $ABCD$ a convex quadrilateral such that each of the segments AB. BC, CD, DA is tangent to Γ. Let Ω be the circumcircle of triangle AIC. The extension of BC beyond A meets Ω at X, and the extension of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T respectively. Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC$$

Another wacky geo problem.



Thoughts: The incircle condition seems to suggest Pitot's Theorem which states that

$$AD + BC = AB + CD$$

So immediately, I think we can do a little swapping action and get a straight line.

$$AD - CD = BC - AB$$

.

The claim is logically equivalent to proving the following:

$$AB + DT + TX + XA = BC + DY + YZ + ZC$$

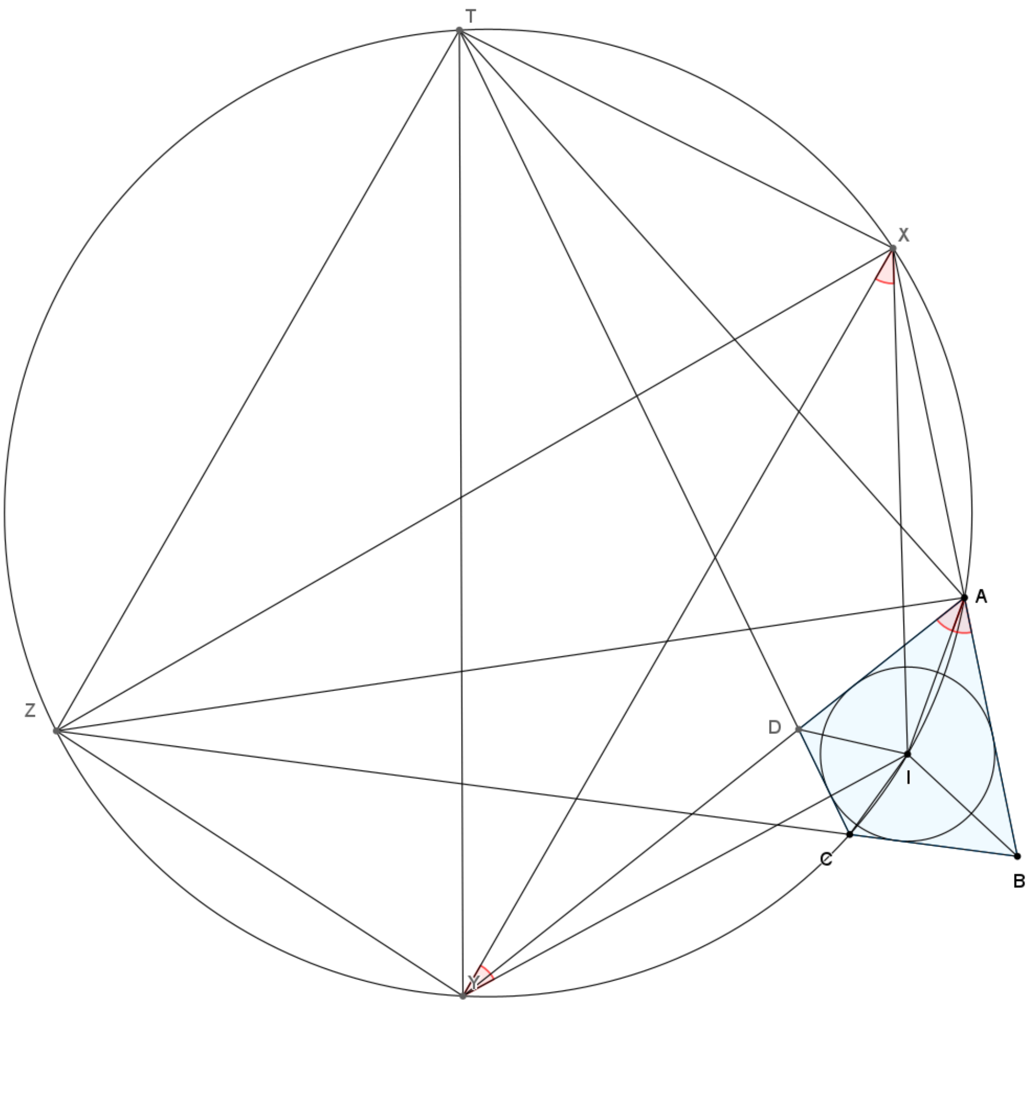
$$DT + TX + BX = DY + YZ + BZ$$

Before we try and interpret this condition let's do some basics.

Claim 1: $IX = IY$ and $IZ = IT$.

So, I was thinking, how do we use this incircle: the incircle gets us this interesting angle condition right, which is actually just an angle bisector of an arc, if we angle chase, we get that the red angles are equal, yay!

$IZ = IT$ follows by similar reasoning.



Claim 2: $TZ \parallel XY$

this follows from the previous claim.

Heuristic: This is enough information to erase I, we can add I back in if we get stuck.

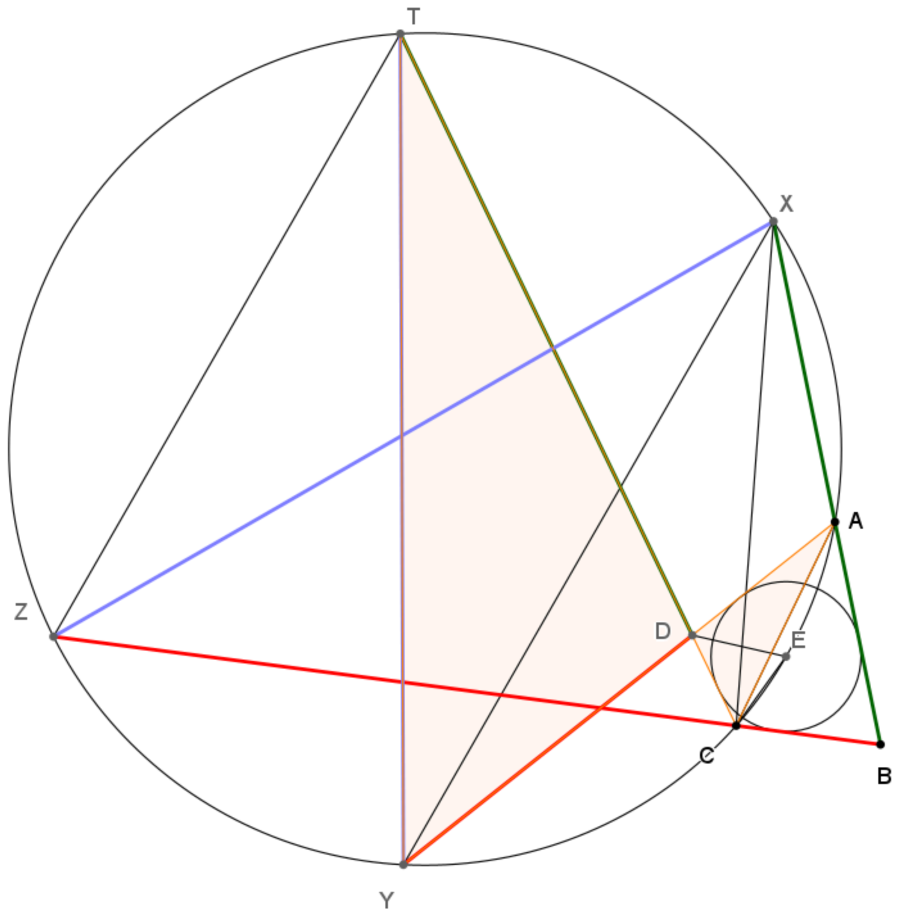
I realise now that **Claim 1** transforms our problem into proving that

$$DT + BX = DY + BZ$$

I was like, hmmm let's deal with these two sides they are annoying, I coloured them blue and oh, they're the same.

Perhaps a more useful form might be

$$DT - DY = BZ - BX$$



Look at this lovely butterfly We have that DAC is mapped to DYT with factor $\frac{AC}{TY}$. We also have that BAC is mapped to BXZ with factor $\frac{AC}{XZ} = \frac{AC}{XZ}$ which means the sides of x and y are the same. *the map would be dilation + reflection but whatever, the ratios are still the same wow*

Which means that $DT - DY = t(DA - DC)$, and $BX - BX = t(BA - BC)$.

Then pitot finishes.

Problem 5: Two squirrels, Bushy and Jumpy, have collected 2021 walnuts for the winter. Jumpy numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circular pattern in the ground around their favourite tree. The next morning Jumpy notices that Bushy had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves. In the kth move, Jumpy swaps the positions of the two walnuts adjacent to walnut k.

Prove that these exists a value of k such that, on the kth move, Jumpy swaps some walnuts a and b such that $a < k < b$.

Assume *FTSOC* that actually Bushy placed the walnuts in such a way that when Jumpy performs the kth move and swaps the walnuts a and b we have that both $a, b > k$ or both $a, b < k$. We will prove that there are an even number of walnuts, this will be a contradiction as # walnuts = 2021. We will say that initially all walnuts are *brown*.

Whenever Bushy performs the kth move we will colour the walnut k *red* if the two walnuts swapped a and b are both $< k$ and *blue* if the two walnuts swapped a and b are both $> k$.

Claim: Now we are going to prove this, after the kth move the following is true:

- every *blue* walnut is not adjacent to any *blue* walnuts
- every *red* walnut is adjacent to two *blue* walnuts.

proof: We prove this by induction, the base case is that this is true after the first move because walnut 1 must be coloured *blue* since there are no two smaller walnuts.

Now assuming that our claim is true after the first $k - 1$ moves on the kth move.

Case 1: a and b are swapped where both a and b are $> k$. k will be coloured blue, and both of k's neighbours are *brown* since they are greater than k and thus have not been coloured yet. Then when we swap them they the colouring is unchanged. So our claim remains true. As the only difference is that a *blue* walnut is added which is not next to any non *brown* walnut

Case 2: a and b are swapped where both a and b are $< k$. k is colored *red* where it was previously *brown*, since a and b are $< k$, we must have performed moves a and b already and thus a and b are both coloured, however neither of they could be coloured red, because they were next to a non *blue* walnut. Thus they must both be *blue*. Swapping the two walnuts does not affect the colouring. So our new *red* walnut is next to two blue walnuts, and all blue *walnuts* are still not next to any other *blue* walnuts.

In both cases our claim remains true on the next move. So by induction, our claim is always true.

At the end all walnuts are coloured either *red* or *blue* and each *blue* walnut is adjacent to two *red* walnuts, and each walnut is adjacent to two *blue* walnuts. So # walnuts is even.

Remark: I checked the n even case and it worked by a trivial construction, so it seemed like an idea of parity was quite important.

It seems that the idea of colouring as Jumpy performs his moves is like the only way to solve the problem? But this makes sense, initially I thought of this, call a walnut "huge" or "tiny" depending on if on the kth move the elements swapped are smaller or larger than k. I wanted to show that every "huge" walnut was adjacent to a "tiny" walnut, it was very difficult to prove this because a sequence of random swaps could mess you up.

So I decided to go bottoms up, I realised that after moves 1 and 2 it was the case that any arrangement of walnuts was reachable as long as 1 and 2 were not adjacent, then it follows that if 3 is also "tiny" then it is not adjacent to either of them, and that if it was "huge" it would have to be adjacent to both 1 and 2, continuing this with a few bigger numbers lead to the idea of colouring as we went along.

Problem 6: Let $m \geq 2$ be an integer, A a finite set of integers (not necessarily positive) and $B_1, B_2, ...B_m$ subsets of A. Suppose that, for every $k = 1, 2, ..., m$, the sum of the elements of B_k is m^k. Prove that A contains at least $\frac{m}{2}$ elements

Remark: wow this solution is very crazy, and dependent on having the bravery that we can think of the problem in terms of base m

What this tells us is that we can write any multiple of m, $0 \leq n < m^{m+1}$ in terms of terms $c_i a_i$ where $a_i \in A$ and $0 \leq c_1 \leq n$.

So we have that we an represent m^m different numbers as this sum.

But this sum can take on at most $(n(m-1))^n$ distinct values where n is the number of elements in A.

This means that $m^m \leq n(m-1)^n$ which implies that $n \geq \frac{m}{2}$.

Remark: I guess m^k is a clue for base m.