IMO 2022 Solutions

Noo~ Way?

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Problem 1: The Bank of Oslo issues two types of coin: aluminum (denoted A) and bronze (denoted B). Marianne has n aluminum coins and n bronze coins arranged in a row in some arbitrary initial order. A chain is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \leq 2n$, Gilberty repeatedly performs the following operation: he identifies the longest chain containing the k^{th} coin from the left and moves all coins in that chain to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the ordering $AABBBABA$ would be $AABBBABA \to BBBAAABA \to AAABBBBA \to BBBBAAAA \to ...$

Find all pairs (n, k) with $1 \leq k \leq 2n$ such that for every initial ordering, at some moment during the process, the leftmost n coins will all be of the same type.

When I refer to chains, I am referring to the maximum length chains, that is a chain which is not a substring of another chain.

We will find all possible values of k for any given n. If $k < n$ then it's possible the first k coins are all A, then the next n coins are B and the final $n - k$ coins are A. In this case Gilberty's operation just doesn't do anything. So $n \leq k$.

Also If $k > \lceil \frac{3n}{2} \rceil$ then consider if we split all the coins up so that there are $\lfloor \frac{n}{2} \rfloor$ A coins, followed by $\lfloor \frac{n}{2} \rfloor$ B coins, then $\lceil \frac{n}{2} \rceil$ A coins and then $\lceil \frac{n}{2} \rceil$ B coins. If $k > \lceil \frac{3n}{2} \rceil$ then we will just repeatedly move each chain to the front, the number of distinct chains will just stay the same, hence we will never have exactly 2 chains, which is when the n leftmost coins are all the same type.

Okay now we show that any k $n \leq k \leq \lceil \frac{3n}{2} \rceil$ will enviably lead to a state where the leftmost n coins will be of the same type. The monovariant to keep in mind is the number of distinct chains.

I'll show that if there is ever 2 chains of the same type, then eventually the number of distinct chains will decrease.

It's impossible to ever move a chain repeatedly to the start, unless we are done. So whenever we move a chain to the start a chain of different type will replace it.

This means if we pick a chain other than the right-most chain to move, then we end up merging a chain with another chain. But, because of the size of k, we are guaranteed to eventually move a chain other than the right-most one. Gilberty must stop eventually, and stops only when we are finished.

Problem 2: Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $x \in \mathbb{R}^+$, there is exactly one $y \in \mathbb{R}^+$ satisfying

$$xf(y) + yf(x) \leq 2.$$

We write $x \sim y$ if and only if $xf(y) + yf(x) \leq 2$. Notice that $x \sim y$ and $x \sim y$ implies that $y = x$.

Suppose that $x \sim y$, we have that by AM-GM $2 \geq xf(y) + yf(x) \geq 2\sqrt{xf(x) \cdot yf(y)}$. $1 \geq xf(x) \cdot yf(y)$. Now we can assume without the loss of generality that this means that $f(x) \leq \frac{1}{x}$. But wait, this would mean that $xf(x) + xf(x) \leq 2$. So actually $x \sim x$ and so $x = y$.

So we have that $f(x) \leq \frac{1}{x}$ for all x and also for all $x \neq y$ we have that

$$xf(y) + yf(x) > 2$$

Remark: doesn't this just scream analysis now.

Assume that for some x we had that $f(x) < \frac{1}{x}$, this must mean that there exists some positive real number C such that $f(x) = \frac{1}{x} - C$ for this particular value of x.

There exists a range of positive real numbers y such that

$$xf(y) + yf(x) \leq \frac{x}{y} + \frac{y}{x} - yC \leq 2$$

This is because the quadratic in terms of y

$$y^2\left(\frac{1}{x} - C\right) - 2y + x$$

has two distinct positive real roots by discriminant. But this contradicts the uniqueness of $x \sim x$ so, actually $f(x) \geq \frac{1}{x}$. So $f(x) = \frac{1}{x}$ for all x.

Remark: This is a very easy p2, The trick is kinda $x \sim x$, but after this the analysis part is very easy.

Problem 3: Let k be a positive integer and let S be a finite set of odd prime numbers. Prove that there is at most one way (up to rotation and reflection) to place the elements of S around the circle such that the product of any two neighbors is of the form $x^2 + x + k$ for some positive integer x.

Visualize this situation as a graph where the primes are the vertices and an edge exists between p and q if pq is a number of the form $n^2 + n + k$. The claim is that in any cycle, there is no alternate cycle.

The key idea is that if there ever exists a node with only two neighbors and those neighbors are also neighbors, then we can remove that node, and induct down.

So we just need to show that such a prime exists, take the largest prime. Because

$$x^2 + x + k \equiv 0$$

has at most two solutions modulo p. The largest prime p must have 2 neighbors.

Now if $qp = n^2 + n + k$ and $rp = m^2 + m + k$, then we have that $qr = (m^2 + n + k)(m^2 + m + k)$

Let's introduce some Algebraic Number Theory. We have that if $\alpha^2 + \alpha + k = 0$ then $n^2 + n + k = \text{Norm}(n - \alpha)$. So then

$$qr = \frac{\text{Norm}((n-\alpha)(m-\alpha))}{p^2}$$

By vieta's formulas $n + m = -1 \bmod p$ and $nm \equiv k$ mod p. Since $n + m + 1 < 2p$ we must have that $n + m = p$.

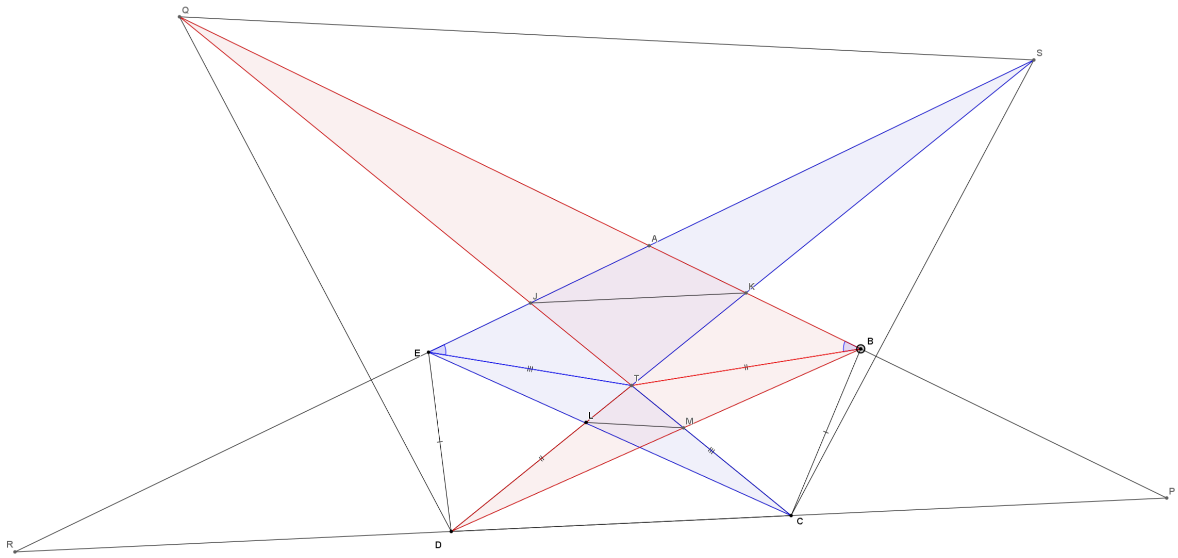
$$qr = \frac{\text{Norm}((nm - k) - (n + m + 1)\alpha)}{p^2}$$

$$qr = \text{Norm}\left(\frac{nm - k}{p} - \alpha\right)$$

Hence qr, crazily enough, is also a number of the form $n^2 + n + k$.

Problem 4: Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume that there is a point T inside $ABCDE$ with $TB = TD$, $TC = TE$ and $\angle ABT = \angle TEA$. Let line AB intersect line CD and CT at points P and Q, respectively. Assume that the points P, B, A, Q occur on their line in that order. Let line AE intersect CD and DT at points R and S, respectively. Assume that the points R, E, A, S occur on their line in that order. Prove that the points P, S, Q, R lie on a circle.

Let J and K be the intersections of AE, CT and AB, DT respectively.



Using the additional angle condition we can deduce that

$$\triangle TES \sim \triangle TBQ$$

Which means that $\angle JQK = \angle JSK$ and hence $(QSJK)$ is cyclic.

$\frac{TS}{TE} = \frac{TQ}{TB}$, $\quad TS \cdot TD = TQ \cdot TC$ this means that by power of a point $(QSCD)$ is cyclic. By Reim's Theorem $CD \parallel JK$ Then by Reim's w.r.t A, we have that $QSPR$ is cyclic.

Problem 5: Find all triplets (a, b, p) of positive integers with p prime and

$$a^p = b! + p$$

First notice that a cannot be 1, as $p > 1$. So RHS > 1.

Now, notice that if $p^2 \mid b!$ then $v_p(\text{RHS}) = 1$ but $v_p(\text{LHS})$ is a multiple of p and hence not 1.

Case 1: $b < p$, it must be then that $b!$ and p are coprime. So $b! + p$ is not divisible by any prime $\leq b$. Which means that $a > b$. So we have that

$$b^p + p < (b+1)^p < \text{LHS} = \text{RHS} = b! + p$$

but clearly $b^p > b!$.

Case 2: $p \leq b < 2p$, we have that $b!$ is a multiple of p and hence a is a multiple of p and so a is at least p^p.

$$p^p \leq a^p = \text{LHS} = \text{RHS} = b! + p \leq p! + p$$

But we have that

$$\begin{aligned} p^p = p^2 \times p^{p-2} &\geq p^2 \times (p-2)! \\ &= (p^2 - 1)(p-2)! + (p-2)! \\ &= (p+1)(p-1)! + (p-2)! \\ &\geq p! + (p-1)! + (p-2)! \\ &\geq p! + p(p-2)! \\ &\geq p! + p \end{aligned}$$

With equality accruing only when $p - 1 = 0$ or 1 because we must have that $(p-2)! = 1$. So $p = 2$ or $p = 3$.

Which yields the solutions $(a, b, p) = (2, 2, 2)$ and $(a, b, p) = (3, 4, 3)$.

Remark: The problems 1, 2, 4 and 5 this year were very easy it seem like.

Problem 6: