IMO 2023 Solutions

Kaguya-san, Miyuki wa sukidesu ka?

Jay Zhao

Problem 1: Determine all composite integers $n > 1$ that satisfy the following property: if d_1, d_2, ..., d_k are all the positive divisors of n with $1 = d_1 < d_2 < ... < d_k = n$, then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \leq i \leq k-2$.

Notice that it must be that

$$d_1, d_2, ..., d_k = \frac{n}{d_k}, \frac{n}{d_{k-1}}, ..., \frac{n}{d_1}$$

Since n is a composite number it must have at least one prime divisor, let p be the smallest such prime divisor and let q be the second smallest prime divisor of n.

d_2 must be the smallest prime divisor of n, as otherwise d_2 is composite and hence a prime divisor of d_2 would be a smaller divisor of n, this is a contradiction.

$$d_2 = p$$

We claim that $d_i = p^{i-1}$ for all $1 \leq i \leq k$. We will do this by strong induction. We have already proven the base case.

Now if our claim is true for i and $i-1$ then $d_i = p^{i-1}$ and $d_{i-1} = p^{i-2}$. If d_{i+1} is composite then it must not have any prime divisors other than p as that would imply there is a a smaller divisor than d_{i+1} which is not one of d_j for $1 \leq j \leq i$ this means that d_{i+1} is a power of p and hence it is p^i as desired.

So we only need to consider when d_{i+1} is a prime other than p. We then have that

$$d_{k+2-i} = \frac{n}{p^{i-2}}, \quad d_{k+1-i} = \frac{n}{p^{i-1}}, \quad \text{and,} \quad d_{k-i} = \frac{n}{q}$$

Where q is some prime number larger than p. Which means we then have that

$$\frac{n}{q} \mid \frac{n}{p^{i-1}} + \frac{n}{p^{i-2}}$$

$$p^{i-1} \mid q + pq \quad , \quad i \geq 2$$

$$p \mid q + pq$$

and hence $p \mid q$ which is a contradiction. So our d_{i+1} must be p^i. Claim is true by *POFMI*.

this means that d_k is a power of p and hence n is a non trivial power of p. Indeed all prime powers satisfy the conditions given in the problem because

$$p^{i-1} \mid p^i + p^{i+1}$$

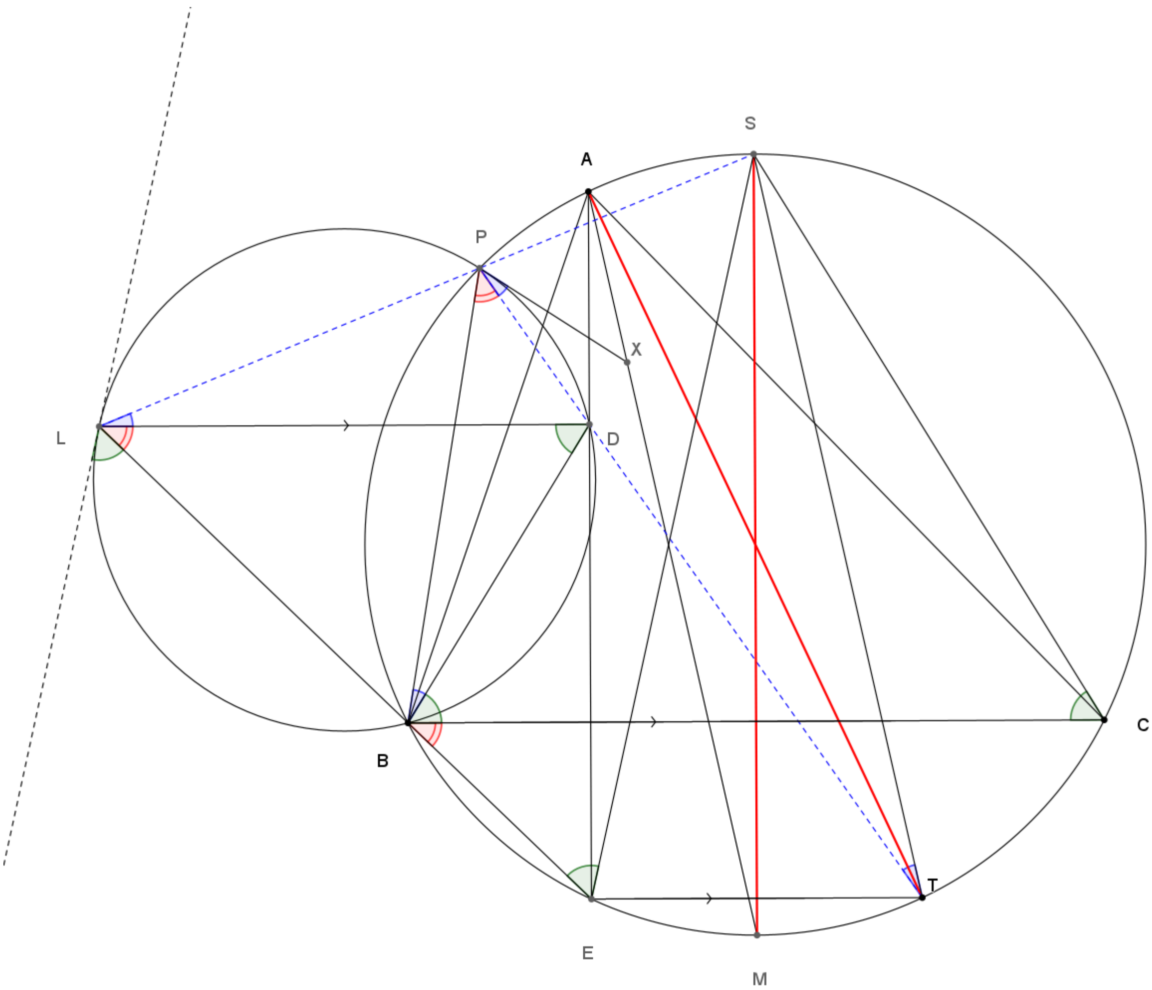
Problem 2: Let ABC be an acute-angled triangle with $AB < AC$. Let Ω be the circumcircle of ABC. Let S be the midpoint of the arc CB of Ω containing A. The perpendicular from A to BC meets BS at D and meets Ω again at $E \neq A$. The line through D parallel to BC meets line BE at L. Denote the circumcircle of triangle BDL by ω. Let ω meet Ω again at $P \neq B$. Prove that the line tangent to ω at P meets line BS on the internal angle bisector of $\angle BAC$

Let M be the midpoint of the arc $\overset{\frown}{AC}$ not containing A. We have that $\overline{SM}$ is diameter of Ω. Let T be the point diametrically opposed to A with respect to the circle Ω.

Notice that ET is parallel to BC because $\angle AET = 90°$ and $ET \perp AE$ and $BC \perp AE$. So we have that T, D, P is collinear by Reim's theorem.

Next we prove that the tangent to ω at L is parallel to SE, this is by angle chasing with the green angles shown below.

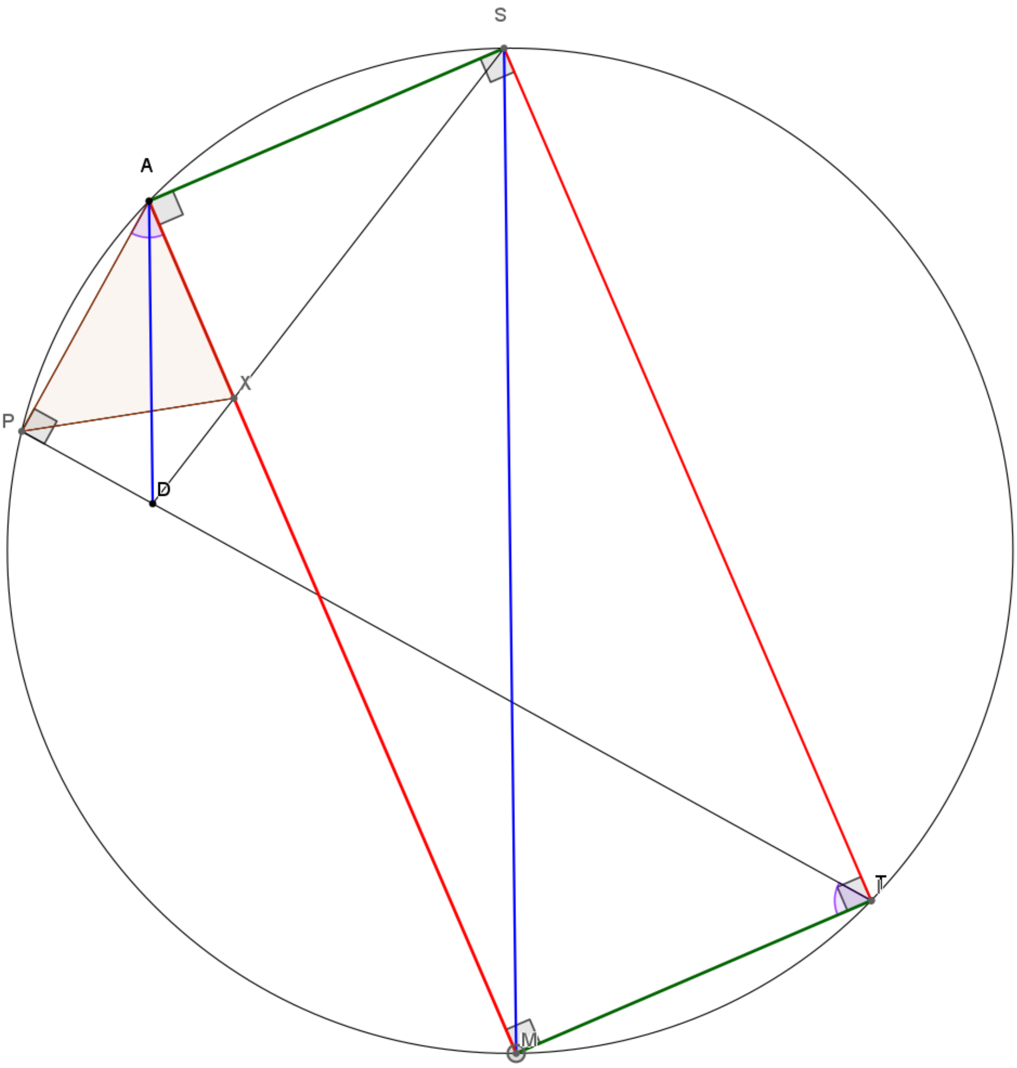
We have that S, P and L are collinear, again, by Reim's theorem.



$$\angle ABD = \angle AES = \angle EAM$$

Hence AM is tangent to the circle (ADB) at A. Notice that DB is the radical axis of ω and (ADB)

Now we let X be the intersection of SB and AI.



Notice also that AX is parallel to ST because both are perpendicular to A S. We have that $(ASTM)$ forms a rectangle.

It is sufficient now to prove that $XA = XP$ as that implies that XP is the tangent of ω.

This is somehow the hard part, this is like fucking impossible, Evan Chen used projective geometry 😭. I'll do a terrible proof, we coord-bash. Yay!

Remark: Isos with right angle = midpoint, midpoint + parallel line = projective. Besides that the problem wasn't very difficult.

I need to get rest.

Oh yeah, I did manage to reduce the problem down to something that seems trivial, when you get this and you can't solve it, try taking a step back.

Problem 3: For each integer $k \geqslant 2$, determine all infinite sequences of positive integers $a_1, a_2, \ldots$ for which there exists a polynomial P of the form $P(x) = x^k + c_{k-1}x^{k-1} + \cdots c_1 x + c_0$, where $c_0, c_1, \ldots, c_{k-1}$ are non-negative integers, such that

$$P(a_n) = a_{n+1}a_{n+2}\cdots a_{n+k}$$

for every integer $n \geqslant 1$.

Note that the polynomial P is strictly increasing over the natural numbers.

Now either the sequence is strictly increasing or it is constant. This is because if $a_{n-1} > a_n$ then we have that $a_{n+k} = \frac{P(n)}{P(a_{n+1})} = a_n$. So $a_{n+k} < a_n < a_{n-1}$. Hence there is some index l such that $a_n > a_{l-1} > a_l$. So we can do infinite descent which is bad.

This is a lower bound on a_{n+1}, let's see if we can find an upper bound.

Notice that in the equality case we have that $a_{n+k} = a_n$ for all n. So we will eventually get a pair of decreasing consecutive terms unless all the terms are constant.

In the constant case it must be that $P(x) = x^k$ for some x which is only the case if all the other terms are 0. So $P(x) \equiv x^k$.

Notice that there is some $Q(x) \equiv (x+d)(x+2d)\cdots(x+kd)$ such that all the terms of $P(x)$ are less than the terms of $Q(x)$ besides $Q(x)$. So we have that if $a_{n+1} > a_n + kd$ then we have that $a_{n+1}a_{n+2}\cdots a_{n+k} \geq (a_n+d)(a_n+2d)\cdots(a_n+kd) > P(a_n)$.

So the relative difference adjacent terms is bounded between 1 and C for some constant C. (This is inclusive).

Oh my god, this means that eventually the relative differences repeat infinitely many times, but then this means we have exactly

$$P(x) = (x+d_1)(x+d_2)(x+d_3)\ldots(x+d_k)$$

for infinitely many values of x and thus

$$P(x) \equiv (x+d_1)(x+d_2)(x+d_3)\ldots(x+d_k)$$

Holy tuple pigeon hole principle. Anyway, this seems like very good progress.

But notice that for the element right after this sequence, we can pigeon hole again, then the next k differences are all the same, this is only true if all the differences are the same. Because we'd have $d_2 - d_1 = d_1$. And so on.

Remark: Wow doing this problem was actually super insightful, the idea of tuple pigeonhole appeared in 2024 IMO P3, I hope it comes up again because wow it's really cool, but it's unlikely >:(. The upper bound is very easy to prove, in fact it was like the first idea I had, the lower bound then should come naturally.

What's funny is that it's very easy to get a lower bound that could be decreasing, and that is good progress too.

The idea reducing a sequence down is very natural once you have that a single equality makes the sequence constant. This is very nice.

Sadly, I had to cheat for the lower bound, but it should come very naturally. I guess the idea is that if you have an upper bound, you should be brave enough to come up with a lower bound.

Also note that in real mathematics, just having a result is really useful, even if it is not tight.

Problem 4:

Let $x_1, x_2, ..., x_{2023}$ be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + ... + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, ..., 2023$. Prove that $a_{2023} \geq 3034$.

So we have that a_1 is 1 and a_2 is at least 3. The first is trivial, the second is because

$$\frac{x_1}{x_1} + \frac{x_2}{x_2} + \frac{x_1}{x_2} + \frac{x_2}{x_1} > 4$$

as we do not have equality.

Then we have that $a_{n+1} \geq a_n$. The reason for this is that

$$a_{n+1}^2 = a_n^2 + \frac{x_{n+1}}{x_{n+_1}} + ... > a_n^2$$

We claim now that $a_{2n} > 3n$. To do this it is sufficient to prove that we cannot have

$$a_{n+1} > a_{n-1} + 3$$

This then finishes off the problem nicely because $a_{2022} \geq 3033$ and then $2_{2023} \geq 3034$.

$$\begin{aligned} a_{n+1}^2 = a_{n-1}^2 &+ \frac{x_n}{x_n} + \frac{x_{n+1}}{x_{n+1}} + \frac{x_n}{x_{n+1}} + \frac{x_{n+1}}{x_n} \\ &+(x_n + x_{n+1})\left(\frac{1}{x_1} + \cdots + \frac{1}{x_{n-1}}\right) \\ &+\left(\frac{1}{x_n} + \frac{1}{x_{n+1}}\right)(x_1 + ... + x_{n-1}) \end{aligned}$$

Then we apply AM-GM on $\frac{x_n}{x_{n+1}} + \frac{x_{n+1}}{x_n} > 2$, and again, apply AM-GM on

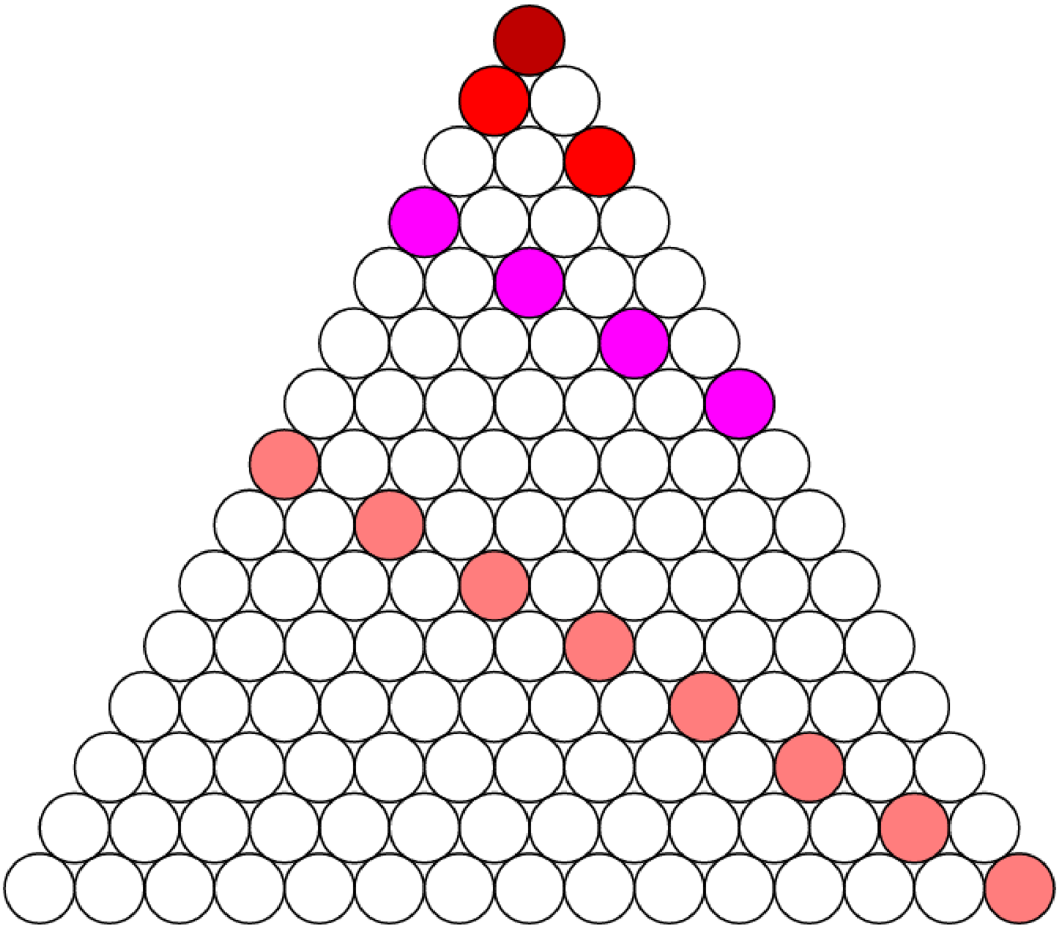
$$x_i\left(\frac{1}{x_1} + ... + \frac{1}{x_{n-1}}\right) + \frac{1}{x_i}(x_1 + ... + x_{n-1}) \geq 2\sqrt{9k^2} = 6k$$

$$a_{n+1}^2 > 9k^2 + 2 + 2 + 12k = (3k+2)^2$$

the strict inequality comes from the unequalness of x_n and x_{n+1}.

Problem 5: Let n be a positive integer. A *Japanese Triangle* consists of $1 + 2 + \cdots + n$ circles arranged in an equilateral triangular shape such that for each $i = 1, 2, \ldots, n$, the ith row contains exactly i circle, exactly one of which is coloured red. A *ninja path* in a Japanese triangle is a sequence of n circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row.

The answer is $k = \lfloor \log_2(n) \rfloor + 1$. The construction is the following pattern.



The idea behind the construction is for each "layer" we color a circle red so that it just just barely dodge the "cone" of the previous circles.

It works because we can only get one of each shade of red circle.

Now let $m(i, j)$ denote the maximum possible number of red circles sub ninja path starting at the top and ending at the ith row at the jth circle.

Let $r(i, j)$ be 1 if the jth circle in the ith row has a red circle in it, let it be 0 otherwise.

We have that $m(i, j) = \max(m(i-1, j), m(m-i, j-1)) + r(i, j)$

We claim that the sum of $m(i, j)$ in row $i = 2^k - 1$ is k. This is true for $k = 1$. We now prove this inductively.

The main claim is that the sum over row $2^k + n$ is at least $k\left(2^k + n\right) + 2n - 1$.

This is because each row adds a single extra red circle, and also there is always at least one circle from the row above which is $k + 1$.

This finished by induction.

Remark: I actually head solved this problem while falling asleep last night, the function is very natural.

Problem 6: Let ABC be an equilateral triangle. Let A_1, B_1, C_1 be interior points ABC such that $BA_1 = A_1C$, $CB_1 = B_1A$, $AC_1 = C_1B$, and

$$\angle BA_1C + \angle CB_1A + \angle AC_1B = 480°.$$

Let $A_2 = \overline{BC_1} \cap \overline{CB_1}, B_2 = \overline{CA_1} \cap \overline{AC_1}, C_2 = \overline{AB_1} \cap \overline{BA_1}$. Prove that if triangle $A_1B_1C_1$ is scalene, then the circumcircles of triangle AA_1A_2, BB_1B_2 and CC_1C_2 all pass through two common points.

Remark: Monster Geo, Wacky Condition.

The problem is the equivalent to proving that the three circles are coaxial, this can be done by showing they have a common radical axis.

Maybe we can find two points such that they have equal power with respect to all three