IMO 2024 Solutions

my only chance at gold is gone :(

Jay Zhao

This is my first IMO and I managed to get a silver medal thanks to Problem 5 trolling everyone there.

The original problems and discussion can be found on the AOPS Collection.

Problem 1: Determine all real numbers α such that, for every positive integer n, the integer

$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor$$

is a multiple of n.

$\alpha = 2k$ where k is a integer.

Let $a = \lfloor \alpha \rfloor$ and $r = \alpha - a$. We have that $0 < r < 1$. Then we have that

$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor = a \times \frac{n(n+1)}{2} + \lfloor r \rfloor + \lfloor 2r \rfloor + \cdots + \lfloor nr \rfloor$$

Case 0: If α is 0 then clearly all values of a that are even work, and all values of a that are odd fail when $n = 2$.

Case 1: If α is even then $a \times \frac{n(n+1)}{2}$ is a multiple of n, and so $\lfloor r \rfloor + \lfloor 2r \rfloor + \cdots + \lfloor nr \rfloor$ is always a multiple of n. We claim that $\lfloor kr \rfloor = 0$ for all k.

This is by induction; The base case is trivial. Now if $\lfloor r \rfloor = \lfloor 2r \rfloor = \cdots = \lfloor (k-1)r \rfloor = 0$ then k divides $\lfloor kr \rfloor$ but also $0 \le \lfloor kr \rfloor < k$. So $\lfloor kr \rfloor$ is also 0. So our claim is true by *POFMI*.

But if we pick an integer $K > \frac{1}{r}$ we get a contradiction, as $Kr > 1$ so then $\lfloor Kr \rfloor \ge 1$.

Case 2: If α is odd then $\lfloor r \rfloor + \lfloor 2r \rfloor + \cdots \lfloor nr \rfloor \equiv a \times \frac{n(n-1)}{2} \equiv \frac{n(n-1)}{2} \mod n$. We now claim that $\lfloor kr \rfloor = k - 1$ for all r.

Again, by induction; The base case is trivial. Now if $\lfloor r \rfloor = 0, \ldots, \lfloor (k-1)r \rfloor = k-2$, Then $\lfloor r \rfloor + \lfloor 2r \rfloor + \ldots \lfloor (k-1)r \rfloor = \frac{(k-1)(k-2)}{2}$. Notice that $\frac{(k-1)(k-2)}{2} \equiv 1 \mod n$. But since we have that $0 \le \lfloor kr \rfloor < k$ it must be that $\lfloor kr \rfloor$ is also $= k-1$. So our claim is true by *POFMI*.

But if we pick an integer $K > \frac{1}{1-r}$ then $K - Kr > 1$ and so $K - 1 > Kr$ and thus $\lfloor Kr \rfloor \ne K - 1$. This is a contradiction.

Remark: This problem is very very easy, it's extremely natural to write α as the sum of a integer and fractional component. The rest is very natural as well.

Problem 2: Determine all pairs (a, b) of positive integers for which there exist positive integers g and N such that

$$\gcd(a^n + b, b^n + a) = g$$

holds for all integers $n \geq N$.

Clearly $(a, b) = (1, 1)$ works. We'll prove it's the only answer.

Case 1: If $M = 2$ then $ab = 1$ and hence $a = 1$ and $b = 1$.

Case 2: If $M > 2$ then Let $M = ab + 1$. Clearly a and M are coprime, and similarly b and M are coprime. So we have that:

- $a \equiv -\frac{1}{b} \mod M$
- $b \equiv -\frac{1}{a} \mod M$

There exists a large $n \equiv -1 \bmod \varphi(M)$ where $n \geq N$. We then have that

$$a^n + b \equiv \frac{1}{a} + b \equiv -b + b \equiv 0 \mod M$$

and also

$$b^n + a \equiv \frac{1}{b} + a \equiv -a + a \equiv 0 \mod M$$

So since M is a common divisor of $a^n + b$ and $b^n + a$ it must be the case that $M \mid g$.

Then consider $a^{n+1} + b$ and $b^{n+1} + a$. We have that

$$M \mid g \mid a^{n+1} + b$$

and thus $1 + b \equiv 0 \mod M$. So by similar reasoning we have the following:

- $a \equiv -1 \mod M$
- $b \equiv -1 \mod M$

But then $ab \equiv 1 \mod M$. But $M = ab + 1$ and so $ab \equiv -1 \quad \bmod M$. This means that

$$-1 \equiv 1 \mod M$$

and thus $M = 1$ or $M = 2$. But neither can be the case as $M > 2$. So there is no such g and N except when $M = 2$.

Remark: I was very sad to have not solved this problem in contest last year, it's literally impossible to solve this problem without the construction $M = ab + 1$. I'm sure at some point I tried $p \mid ab \pm 1$. I guess I thought that primes would be easy to work with, but they aren't actually.

I think the main reason I didn't solve this problem is because I came up with a lot of reasons why specific cases didn't work. But actually the solution is very general.

Problem 3: Let $a_1, a_2, a_3, ...$ be an infinite sequence of positive integers, and let N be a positive integer. Suppose that, for each $n > N$, a_n is equal to the number of times a_{n-1} appears in the list $a_1, a_2, ..., a_{n-1}$.

Prove that at least one of the sequence $a_1, a_3, a_5, ...$ and $a_2, a_4, a_6, ...$ is eventually periodic.

We transform the problem in the following way, we assign a pile for each positive integer, and initially all the piles are empty ($t = 0$). At $t = n$, we add a stone to pine a_n. We say that the stones that are added at $t \leq N$ are bedrock, and the stones added at $t > N$ are jade. We will have the piles in size order where pile 1 is the left-most pile.

We say that a pile A increases the height of another pile B if at some time a stone is added to pile A and then at the next time-step a stone is added to pile B.

Let M be the number of the right-most pile containing bedrock. Let h_k denote the number of stones in pile k at some given time. h_k is an implicit function of time t.

1. If $h(k+1)$ gets arbitrarily tall over time, then $h(k)$ gets arbitrarily tall over time, and there exists a constant C such that $h(k) > h(k+1) - C$ at all times.

   proof: For every purely jade pile of stones, every time the stack reaches a height of $k+1$ is must reach a height of k first. There are finitely many piles of stones containing bedrock, but each of them can increase the height of pile $k+1$. So the height of pile $k+1$ can increase finitely many times more than pile k.

2. The piles with numbers greater than M can have at most M stones in them.

   proof: Assume otherwise, then consider the first time we add a stone to a pile $> M$ and it's height becomes greater than M, it must be that on the time step before there where at least $M+1$ piles with height $> M$ contradicts this being the first time.

3. Some pile must become arbitrarily tall, assuming otherwise there are finitely many new piles created, but each time this happens we add a stone to pile 1, contradiction.

4. Let L be the number of the right-most pile which grows to be arbitrarily tall + 1, we claim that we eventually alternate between adding stones to piles $< L$ and piles $> M$. This is because eventually all the stones in piles between L and M inclusive will be added to for the last time, then every time we add to a pile $> M$, since the pile has height $< M$, we must then add to a pile $< M$ and hence $< L$. and eventually all the piles of height $< L$ will reach a height $> M$.

   We also have that past some time t the new pile we create will reach a height of $L - 1$.

5. The key idea now is to notice that if we consider only the subsequence containing the piles $< L$. The next term in the sequence is dependent only on the relative heights of the "small" piles and a_n. But there are finitely many combinations of this.

This is because there are finitely many values of a_n that is $< L$. Then we have an upper bound $h(k) > h(k+1) - C$, to finish we just need a lower bound.

The idea is that if the first k piles are super super big, the they increase off to infinity on their own. And so if a single pile is way too big, then it drags all the piles before it up. So any pile can only be so much bigger than the left-most pile $< L$.

So the relative differences is bounded. We are done.

Remark: It's pretty interesting that we never proved what the form of the periodicity actually is, it might be $1, 2, 3, 1, 2, 3, 1, 2, 3, ...$ or it might even be $1, 1, 1, 1, 1$ or something wacky like $1, 4, 5, 6, 3, 2, ...$ But what's crazy is that it doesn't matter. Pigeonhole = Periodicity. This is super cool.

The motivation for trying this might be noticing an upper bound, it then makes sense to want to prove a lower bound, but without the insight to do this we might not be brave enough to even try and prove this.

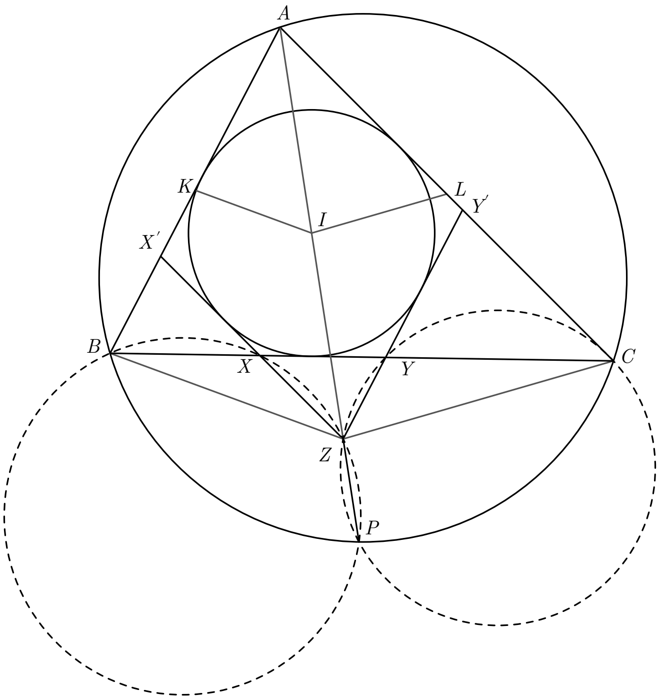
But if we have in mind that Pigeonhole = Periodicity, then it becomes motivated to try and find a lower bound.

In fact, I think proving an upper bound is good motivation to prove a lower bound, sometimes it's good to prove that such a upper / lower bound even exists, especially in a combinators problem like this one where the numbers themselves don't actually matter all that much.

Nonetheless, this problem is very natural once you know the solution.

Problem 4: Let ABC be a triangle with $AB < AC < BC$. Let the incenter and incircle of triangle ABC be I and ω, respectively. Let X be the point on line BC different from C such that the line through X parallel to AC is tangent to ω. Similarly, let Y be the circumcircle of triangle ABC at $P \neq A$. Let K and L be the midpoints of AC and AB, respectively. Prove that $\angle KIL + \angle YPX = 180°$.

Let the tangents described intersect at Z. Extending the tangent containing X and Y to meet AB and AC at X' and Y'.



We find that $AX'ZY'$ is a parallelogram, and furthermore it is a rhombus, so by symmetry A, I and T are collinear.

Because of this $AZ = 2AI$ Which means that Z is the reflection of A over I.

$\angle KIL = \angle BZC$ by a dilation of factor 2 centred at A.

By Reim's theorem we have that $(BXZP)$ and $(CYZP)$ are cyclic. But we can also just angle chase.

Finally to finish off

$$\angle KIL + \angle YPX = \angle BZC + \angle XPZ + \angle ZPY = \angle BZC + \angle XBZ + \angle ZCY = 180°$$

Problem 5: Turbo the snail plays a game on a board with 2024 rows and 2023 columns. There are hidden monsters in 2022 of the cells. Initially, Turbo does not know where any of the monsters are, but he knows that there is exactly one monster in each row except the first row and the last row, and that each column contains at most one monster.

Turbo makes a series of attempts to go from the first row to the last row. On each attempt, he chooses to start on any cell in the first row, then repeatedly moves to an adjacent cell sharing a common side. (He is allowed to return to a previously visited cell.) If he reaches a cell with a monster, his attempt ends and he is transported back to the first row to start a new attempt. The monsters do not move, and Turbo remembers whether or not each cell he has visited contains a monster. If he reaches any cell in the last row, his attempt ends and the game is over.

Determine the minimum value of n for which Turbo has a strategy that guarantees reaching the last row on the n-th attempt or earlier, regardless of the locations of the monsters.

The answer is 3. First we will show we cannot do any better. It's possible that the first time Turbo moves down to second row he lands on a monster and gets gobbled up instantly. Then Turbo will respawn. Later when Turbo moves down to the third row for the first time it's possible there is a monster in the third row where Turbo moved down to. So Turbo cannot guarantee make that he will make it to the end in at least 3 attempts.

Turbo should visit every cell in the second row. To do this, move to some column in the first row, move down, then move back up and repeat. It's fine if Turbo dies here, we just repeat until Turbo has visited every cell in the second row. Turbo die exactly once.

Case 1: Turbo dies on a cell not on the edge. We now attempt to sidestep the monster.

We move to the side of the monster, step down one, step in front of the monster then continue down till the end.

Case 2: Turbo dies on a cell on the edge. Turbo now steps to the side of the monster and travels down in a staircase path.

If Turbo dies moving down a row, then we retrace the path two steps before we died. Move down by one, the move all the way back to the edge we died on originally. Then travel all the way down.

If Turbo dies moving across a row, then we retrace the path to right before we died, then again move all the way back to the edge we died on originally. Then travel all the way down.

Turbo finishes in 3 attempts, if Turbo doesn't die along the staircase, he finishes in 2

Problem 6: A function $f : \mathbb{Q} \to \mathbb{Q}$ is called *aquaesulian* if the following property holds: for every $x, y \in Q$,

$$f(x + f(y)) = f(x) + y \quad \text{or} \quad f(f(x) + y) = x + f(y).$$

Show that there exists an integer c such that for any aquasulian function f there are at most c different rational numbers of the form $f(r) + f(-r)$ for some rational number r, and find the smallest possible value of c.

The answer is $c = 2$. In any aquasulian function, there are at most 2 possible values of $f(r) + f(-r)$, this we will prove later. c cannot be any less than 2 because of the function $f(x) = \lfloor x \rfloor - \{x\}$.

Assume that f is an aquasulian function, we are now going to prove facts about f.

1. For every rational value of x $f(x + f(x)) = x + f(x)$. This is by $P(x, x)$
2. f is injective.

   proof: If $f(a) = t = f(b)$ then assume *WLOG* that $f(a + f(b)) = b + f(a)$

   $$a + f(a) = f(a + f(a)) + f(a + f(b)) = b + f(a)$$

   so $a = b$. Therefore f is injective.
3. Given that $f(a + f(b)) = f(a) + b$ it must be that either

   $$f(b) + f(-b) = 0 \quad \text{or} \quad f(b) - f(-b) = 0.$$

   proof: This is because $P(a + f(b), -b)$ implies that

   $$f(a + f(b) + f(-b)) = f(a) \quad \text{or} \quad f(f(a)) = a + f(b) - f(-b)$$

   We can use injectivity for the first condition and rearrange the second.
4. Let S be the set $S = \{x \mid f(x) + f(-x) \neq 0, x \in \mathbb{Q}\}$

   $$x \in S \Rightarrow f(f(x)) - x = f(x) + f(-x)$$
5. For $x, y \in S$ we have that $f(x) + f(-x) = f(y) + f(-y)$

proof: Assume without loss of generality that $f(x + f(y)) = y + f(x)$. We then have that

$$f(y) + f(-y) = f(f(x)) - x = f(x) + f(-x))$$

So either $f(x) + f(-x) = 0$ or $f(x) + f(-x) = t = f(x_0) + f(-x_0)$ for some $x_0 \in S$.

Fact 5 implies that there can be at most 2 values of $f(r) + f(-r)$ we are done.

Remark: This problem is actually not very difficult, proving injectivity is rather easy. We can represent this problem as a directed graph. The idea is that if a number is pointed to already and we get it again, we can use injectivity to rule out one of the "or"s. I ended up doing $P(a + f(b), x)$ and then notices that $-b$ just works.