Mock 1

no cheeky comment this time

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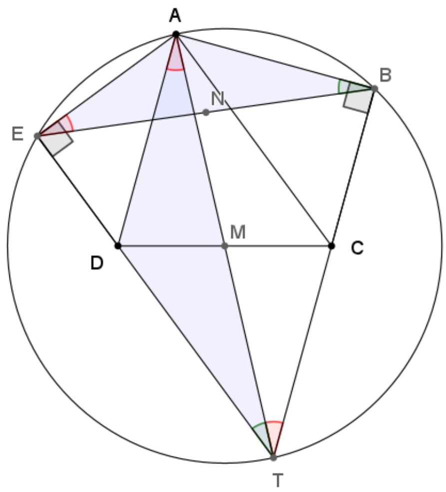
Problem 1: Let $ABCDE$ be a convex pentagon such that $\angle ABC = \angle AED = 90°$. Suppose that the midpoint of CD is the circumcentre of triangle ABE. Let O be the circumcentre of triangle ACD.

Prove that line AO passes through the midpoint of segment BE.

Let ED and BC intersect at point T, by Thales theorem, it must be the case that AT is the diameter of the circumcircle of triangle ABE. Hence A, M and T are collinear.



Claim 1: $\triangle ADT \sim \triangle EAB$

$$\angle ATD = \angle ABE, \quad \text{by cyclic quads}$$

$$\angle DAT = \angle CTA = \angle BEA, \quad \text{parallel lines + cyclic quad}$$

Then our claim is true by AA.

Let N be the midpoint of segment BE.

Claim 2: $\angle DAN = 90° - \angle ACD$

$$\angle DAN = \angle EAN - \angle EAD$$

By similar triangles $\angle EAN = \angle ADM$. $\angle EAD = 90° - \angle EDA = 90° - \angle DAC$. Hence

$$\angle DAN = \angle ADN - 90° + \angle DAC = 180 - \angle ADC - 90° = 90° - \angle ACD$$

Claim 3: $\angle DAO = 90° - \angle ACD$.

Let H be the orthocenter of triangle ACD, we have that $\angle CAH = 90° - \angle ACD$. Since H and O are isogonal conjugates with respect to triangle ACD it must be that also,

$$\angle DAO = 90° - \angle ACD$$

Claim 2 and **Claim 3** together imply that A, N and O.

Q.E.D.

Problem 2: Let $n > 1$ be an integer. Determine the smallest positive integer d for which there exists $2n$ positive integers $a_1, a_2, ..., a_n$, $b_1, b_2, ..., b_n$ such that the $n + 1$ products

$$a_1 a_2 a_3 ... a_n,$$
$$b_1 a_2 a_3 ... a_n,$$
$$b_1 b_2 a_3 ... a_n,$$
$$\vdots$$
$$b_1 b_2 b_3 ... b_n,$$

form, in that order, a strictly increasing arithmetic progression with common difference d.

Let d be the smallest possible d for which there existed such a sequence. Assume that out of all the possible pairs of sequences (a_i) and (b_i) that the two sequences $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ were such that $a_1, a_2, a_2 ... a_n$ has the smallest possible value. Such a minimal sequence must exist because each of a_i is a positive integer and so the set of possible values of $a_1 a_2 a_2 ... a_n$ is well ordered.

Claim 1: a_i and b_i must be coprime for all $1 \leq i \leq n$.

Assume otherwise, then let $g \geq 1$ be the greatest common divisor of a_i and b_i for some i. Consider the pair of sequences $a_1, ..., a_{i-1}, \frac{a_i}{g}, a_{i+1}, a_n$ and $b_1, ..., b_{i-1}, \frac{b_i}{g}, b_{i+1}, ... b_n$. The $n + 1$ products formed by this new pair of sequences is equal to the corresponding $n + 1$ products of the original sequences, just that each product is now smaller by a factor of g. This contradicts the minimality of our two original sequences. So our claim is true.

Now let $a := a_1 a_2 a_3 ... a_n$.

Claim 2: $\frac{b_i}{a_i} = \frac{a+id}{a+(i-1)d}$.

We know that $b_1 ... b_{i-1} a_i a_{i+1} ... a_n = a + (i - 1)d$ and $b_1 ... b_i a_i ... a_n = a + (id)$. Dividing the latter by the former gives the desired result.

Claim 3: $a_i = \frac{a+(i-1)d}{(a,d)}$ and $b_i = \frac{a+id}{(a,d)}$ where (a, d). Where $(a, d) = \gcd(a, d)$.

$a_i = \frac{a+(i-1)d}{g}$ where g is the greatest common divisor of $a + (i - 1)d$ and $a + id$. But notice that by euclidean algorithm, the greatest common divisor of those two is the greatest common divisor is (a, d).

So we have that a_i forms an arithmetic progression and b_i forms an arithmetic progression where $a_{i+1} = b_i$.

So we have that $d = (b_1 - a_1)(a_2 a_3 ... a_n)$ where $b_1 - a_1$ is at least 1 ass $b_1 > a_1$ and $a_2 a_3 ... a_n$ is at least $2 \times 3 \times ... \times n = n!$.

$d = n!$ and this is achieved with the sequence $a_i = i$ and $b_i = i + 1$.

Q.E.D.

Problem 3: Ross has 2023 treasure chests, all of which are unlocked and empty at first. Each day, Ross adds a new gem to one of the unlocked chests of his choice, and afterwards, a fairy acts according to the following rules:

- if more than one chests unlocked, it locks one of them, order
- if there is only one unlocked chest, it unlocks all the chest.

Given that this process goes on forever, prove that there is a constant C with the following property: Ross can ensure that the difference between the numbers of gems in any two chests never exceeds C, regardless of how the fairy chooses the chests to lock.

Problem 4: Let $m \geq 2$ be an integer, A a finite set of integers (not necessarily positive) and B_1, B_2, ...