# Random Walks on Graphs

the only cool thing I learnt from MATH199 at UOA

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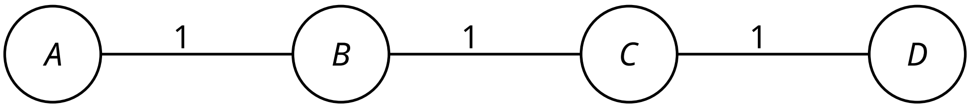
Say we have a undirected graph $G = (V, E, w)$ with vertices $V$ and edges $E$ denoted $(a, b)$ and where each edge has a weight positive real weight given by $w(a, b)$.

We will consider $w(a, b) = 0$ if there is no edge between $a$ and $b$, and we will let $d(a) = \sum w(a, b)$ be the sum of the weights of all the edges connected to $a$. *In the case where all the weights are* 1 *this is the same as the degree of* $a$.

We'll say a random walk is a process that begins at some vertex, and at each time step we move to another vertex randomly. If at a given time we are at vertex $a$, we will move to vertex $b$ with probability $\frac{w(a,b)}{d(a)}$. So the probability that we move to a vertex is proportional to the weight of the vertex that we'll have to move through.

Lets consider this following graph for now:




If we start at vertex $A$ then there is a probability of 1 that we move from vertex $A$ to vertex $B$ at the next time step.

Then once we are at vertex $B$ there is a $\frac{1}{2}$ probability that we will move from $B$ to $A$ at the next time step and a $\frac{1}{2}$ probability that we will move from $B$ to $C$ instead in that same time step.

Let $p_t$ denote the probability distribution at time $t$, that means $p_t(a)$ is the probability that after $t$ time steps we are at vertex $a$.

Clearly we have that

$$\boldsymbol{p}_{t+1}(a) = \sum \frac{w(a, b)}{d(b)} \boldsymbol{p}_t(b)$$

We can represent this situation using a markov chain. Let $\boldsymbol{M}$ be a matrix where $\boldsymbol{M}(i, j)$ has the value $w(i, j)$. Let $\boldsymbol{D}D$ be a diagonal matrix with $\boldsymbol{D}(i, i) = d(i)$.

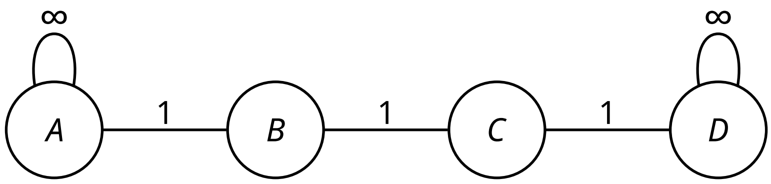
We can write

$$\boldsymbol{p}_{t+1} = \boldsymbol{M}\boldsymbol{D}^{-1}\boldsymbol{p}_t$$

For a walk starting at position $A$ we can say $\boldsymbol{p}_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}$.

$$\boldsymbol{p}_t = \left(\boldsymbol{M}\boldsymbol{D}^{-1}\right)^t \boldsymbol{p}^0$$

Now consider this following graph




What this graph represents is a situation where if we are at $A$ then on the next time step we stay at $A$, and if are at $D$ then at the next time step we stay at $D$.

Let $\boldsymbol{p}(a)$ denote the probability that a random walk reaches $A$. As our walks are all infinite processes, we should expect two long term outcomes: either a walk reaches $A$ and gets stuck there or a walk reaches $D$ and gets stuck there.

Well consider vertex $B$. There is a $\frac{1}{2}$ probability that we go straight to $A$ and a $\frac{1}{2}$ probability that we go to $C$. So $\boldsymbol{p}(B) = \frac{1}{2}(1) + \frac{1}{2}\boldsymbol{p}(\boldsymbol{C})$.

More generally we have that

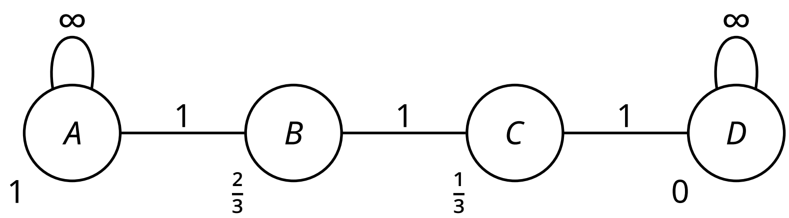
$$\boldsymbol{p}(a) = \sum \frac{w(a,b)}{d(a)}\boldsymbol{p}(b)$$

Which we can represent as

$$\boldsymbol{D}\boldsymbol{p} = \boldsymbol{M}\boldsymbol{p} \text{ or } \boldsymbol{p} = \boldsymbol{D}^{-1}\boldsymbol{M}\boldsymbol{p}$$

So the vector $\boldsymbol{p}$ is an eigenvector of the transition matrix $\boldsymbol{D}^{-1}\boldsymbol{M}$ with eigenvalue 1.

Solving for the eigenvector with $\boldsymbol{p}(A) = 1$ and $\boldsymbol{p}(D) = 0$ we get that. $\boldsymbol{p}(B) = \frac{2}{3}$ and that $\boldsymbol{p}(C) = \frac{1}{3}$.




This is quite nice, it looks like the probability of reaching $A$ decays linearly. I claim that if we interpret $\boldsymbol{p}$ to represent potential in the context of a circuit, this will also make sense.

The vertices will be points on our circuit, and the edges will be wires / resistors connecting them. Then a random walk can be interpreted as the path an electron take along the circuit, starting at $A$ with potential 1 and ending at $D$ with potential 0.

We will represent the resistance of each node to be $R(a,b) = \frac{1}{w(a,b)}$. In this way $w(a,b)$ kinda represents the conductance.

We can then apply ohm's law to define what should be "current". $I(a,b) = \frac{\boldsymbol{p}(a)-\boldsymbol{p}(b)}{R(a,b)}$.

Kirchkoff's voltage laws obviously hold, if you can't see this please ponder it.

We will show that Kirchkoff's current law's must hold,

$$\begin{aligned}\sum_{b\in N(a)} I(b,a) &= \sum_{b\in N(a)} \frac{\boldsymbol{p}(b)-\boldsymbol{p}(a)}{R(a,b)} \\ &= \sum_{b\in N(a)} w(a,b)\boldsymbol{p}(b) - \boldsymbol{p}(a)\sum_{b\in N(a)} w(a,b) \\ &= \sum_{b\in N(a)} w(a,b)\boldsymbol{p}(b) - d(a)\boldsymbol{p}(a) \\ &= 0\end{aligned}$$

The total current (signed) going out of every vertex is 0. Our interpretation of voltage, current and resistance provides a valid circuit satisfying Ohm's Law, and both of Kirchkoff's Laws.

In our diagram we'll denote $D$ to be the source and $A$ to be the destination. The probability that starting at the source, taking a single step out and going to the destination before we return to the source we will call the probability of escape.

We can then consider the entire diagram to be one big resistor setup.

This is equal to

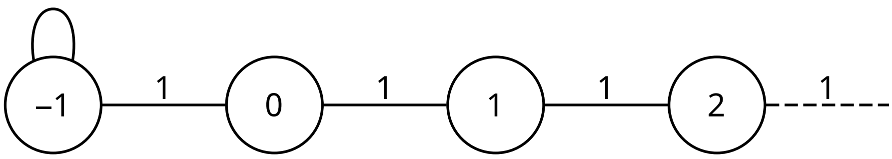
$$\sum_{b\in N(D)} \frac{w(D,b)}{d(D)}\boldsymbol{p}(\boldsymbol{b}) = \frac{\sum_{b\in N(D)} I(b,D)}{d(D)}$$

In a physical sense this is the out-going current at the source. This must be equal to the current going into the source which must be the current across the entire circuit.

$$p_{\text{escape}} = \frac{I_{\text{circuit}}}{d(\text{source})} = \frac{1}{d(\text{source})R_{\text{effective}}}$$

In this way we can transform any question about random walks on a graph into a physics question to do with circuits.

Consider now the following circuit




If we let the source be $-1$ and the destination be $N$ then the effective resistance is $N+1$. Taking $N$ to be arbitrarily large, we have that the probability of escape approaches 0. Since potential decreases linearly, if we let the potential at $-1$ be 1 and the potential at $N$ be 0 then the potential represents the probability that a walk reaches $-1$ before $N$. The current across the whole circuit can be found to be $\frac{1}{N+1}$.

Then the potential difference between −1 and 0 is given by $1 \times \frac{1}{N+1}$. Hence $\boldsymbol{p}(0) = 1 - \frac{1}{N+1}$ which approaches 1 as $N$ becomes really big.

In the long term there are two possibilities, a walk goes out to arbitrarily large values of $N$ or it travels to −1 and gets stuck. But the probability that a walk goes out to infinity is 0. So a random walk starting at 0 will reach −1 with probability 1.

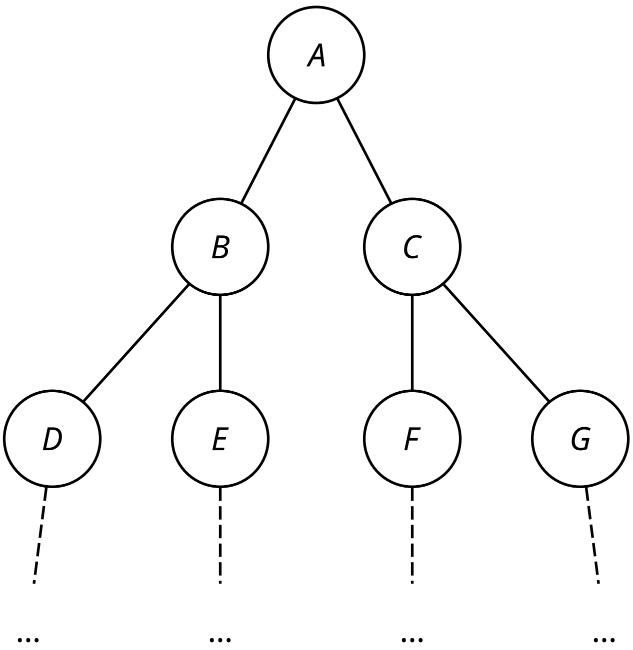
We can connect this to the catalan numbers. The probability that a path gets trapped at −1 on the $2n + 1$th move is $\frac{C_n}{2^{2n+1}}$.

This means that

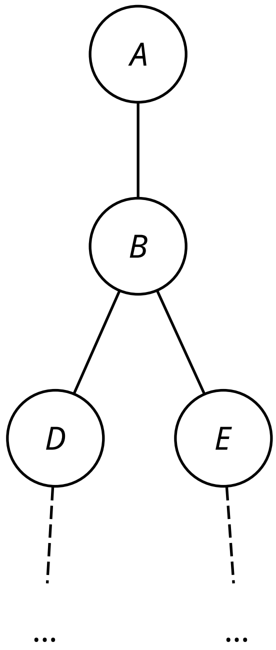
$$\sum \frac{C_n}{4^n} = 2$$

which is kinda a crazy result.

Let's take a look at a binary tree




Let's first compute the resistance of the whole circuit. Since the branches going $A \to B$ is symmetric with the branch going $A \to C$ The resistance of this whole circuit is equal to half the resistance of the following circuit.




The resistance of this sub-circuit is $1 + R$ where $R$ is the resistance of the whole circuit. So we have that $R = \frac{1+R}{2}$ and hence $R = 1$.

We then have that $d(A) = 2$ We then have that

$$p_{\text{escape}} = \frac{1}{Rd(A)} = \frac{1}{2}$$