Category Theory

I love pretty diagrams

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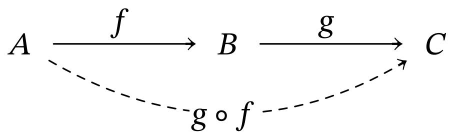
1 Introduction

1.1 Categories

A category consists of a class of objects, a set of arrows between pairs of objects called morphisms, and two operations: composition and identity.

If $f \in \hom(A, B)$ and $g \in \hom(B, C)$, then their composition $g \circ f$ is in $\hom(A, C)$. You might also see me write if $f : A \longrightarrow B$ and $g : B \longrightarrow C$ then $g \circ f : A \longrightarrow C$.



And for every $A \in \mathrm{ob}(C)$ there is a identity morphism $\mathrm{id}_A : A \longrightarrow A$. Where id_A is the unique morphism such that for any $f : A \longrightarrow B$, $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$.

> *Example*: We can consider a group as a category with a single object where the morphisms represent elements of the group and composition is given by the group operation. This is denoted BG where G is the group.

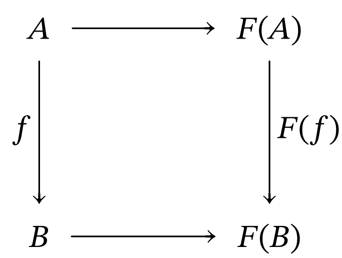
Remark: This is different from the category of groups which is denoted Grp.

A morphism $f : A \longrightarrow B$ is a **isomorphism** if there exists f^{-1} where $f^{-1} \circ f = \mathrm{id}_A$ and where $f \circ f^{-1} = \mathrm{id}_B$.

The category C^{op} is the opposite category of C where all arrows are reversed.

1.2 Functors

A functor is a structure preserving map between categories. A functor $F : C \longrightarrow D$ maps every object in C to an object in D, and maps every arrow in C to an arrow in D such that composition is preserved and identities and sent to identities.



Example: A functor between two groups seen as categories is a group homomorphism.

Example: Consider the set of vector spaces over $\mathbb{F}$ where morphisms are linear transformations. Consider then the functor $F : V \longrightarrow V^*$ mapping a vector space V to it's dual space.

Consider a linear transformation $f : V \longrightarrow W$ where W is another vector space. Typically we define $f^* : W^* \longrightarrow V^*$ as the dual map induced by f.

However this maps a function $f : V \longrightarrow W$ to a function $F(f) : W^* \longrightarrow V^*$. We call this a covariant functor.

Covariant functors are functors that preserve the direction of arrows. If $f : A \longrightarrow B$ is a morphism in C, then $F(f) : F(A) \longrightarrow F(B)$ is a morphism in D. Covariant functors are denoted $F : C \longrightarrow D$.

Contravariant functors are functors that reverse the direction of arrows. If $f : A \longrightarrow B$ is a morphism in C, then $F(f) : F(B) \longrightarrow F(A)$ is a morphism in D. Contravariant functors are denoted $F : C^{\mathsf{op}} \longrightarrow D$.

1.3 Epic and Monic Maps

Monic A map $X \xrightarrow{f} Y$ is monic (a monomorphism) if $f \circ g = f \circ h \implies g = h$ this generalizes the idea of a function being injective.

Epic A map $X \xrightarrow{f} Y$ is epic (an epimorphism) if $g \circ f = g \circ f \implies g = h$

1.4 Natural Transformations

Given two functors F, G from $C \longrightarrow D$, a natural transformation $\eta : F \implies G$ consists of a family of morphisms $F(x) \xrightarrow{\eta_x} G(x)$ for each $x \in C$ where the following square commutes.

$$\begin{array}{ccc} F(x) & \xrightarrow{\eta(x)} & G(x) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(y) & \xrightarrow[\eta(y)]{} & G(y) \end{array}$$

$G(f) \circ \eta_x = \eta_y \circ F(f)$. Natural transformations transform morphisms in a way that respects the underlying structure of the categories and functors involved.

If all of the components η_x are isomorphisms, then we say that the natural transformation is a **natural isomorphism**.

This can be thought of as a change in perspective. Instead of viewing the functors as separate entities, we can see them as different ways of looking at the same underlying structure. So for example we can view v and v^{**} as it turns out these are naturally isomorphic for finite dimensional vector spaces. So we can view a vector space as being naturally isomorphic to its double dual. the same.

2 Universal Constructions

A **universal construction** is a way of describing objects in terms of maps in such a way that it defines the object up to unique isomorphisms.

Initial Objects An initial objects in a category C is an object I such that for all $X \in \text{ob}(C)$ there exists a unique morphism $I \longrightarrow X$.

These are unique up to unique isomorphism because if I and I' are both terminal objects then there exists unique morphisms $f : I \longrightarrow I'$ and $g : I' \longrightarrow I$. Then $f \circ g = \text{id}_I$ and $g \circ f : I' \longrightarrow I'$ and hence must be $\text{id}_{A'}$.

Terminal Objects A terminal object in a category C is an object T such that for all $X \in \text{ob}(C)$ there exists a unique morphism $X \longrightarrow T$.

Again these are unique up to unique isomorphism.

Products A product of two objects A and B in a category C is an object $A \times B$ together with two projection morphisms $\pi_A : A \times B \longrightarrow A$ and $\pi_B : A \times B \longrightarrow B$ such that for any object X with morphisms $f : X \longrightarrow A$ and $g : X \longrightarrow B$, there exists a unique morphism $h : X \longrightarrow A \times B$ such that the following diagram commutes.