Calculus 101

introduction to real analysis

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1 Real Numbers

☐ **Theorem 1.1** ($\mathbb{R}$ is complete): A sequence converges to a point if and only if it is a Cauchy sequence.

Proof: Any Cauchy sequence has a bounded subsequence. Now consider the set $S = \{x \mid x \geq a_n$ for all $n > N$ some N$\}$. This set has a supremum which the sequence must converge to. ∎

Remark: But we still need to prove that there is a supremum!

An **upper bound** for S is a real number M where $x \leq M$ for all $x \in S$. A **lower bound** is defined similarly.

If a subset of S is bounded from both above and below it is called **bounded**

☐ **Theorem 1.2**: If S is bounded from above it has a least upper bound. If S is bounded from below it has a greatest lower bound.

These are called the supremum and infimum of S.

Proof: The real numbers are defined as Dedekind cuts of the rationals. Which is a partition of the rationals into two non-empty complementary sets A and B where all elements of A are less than all elements of B and A has no largest elements. To take the supremum of a set S, we take the union of all the left cuts. ∎

2 Limits and Series

☐ **Theorem 2.1** (Monotone Convergence Theorem): A monotonic increasing sequence which is bounded above converges.

Proof: Let L be the supremum of the sequence. For any $\varepsilon > 0$ there is some a_n for which $L - \varepsilon < a_n < L$ as otherwise the supremum would not be minimal. Thus the sequence converges to L. ∎

Limit supremum and infimum

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_n, \ldots\}$$

The limit supremum of a sequence a_n is the limit of the supremum of the tails of the sequence. Or what the supremum will eventually be after we throw away terms.

We allow the supremum to be $+\infty$ is the sequence is not bounded from above.

☐ **Theorem 2.2**: If a sequence is bounded from above then it's limit supremum is finite.

Proof: The sequence of suprema is non increasing and bounded from above so it converges by ☐ Theorem 2.1. ∎

The limit infimum is defined similarly.

Convergence of Series A series converges if the sequence of it's partial sums converge.

Remark: You should already know this one from 250. Anyway, what's nice about this definition is that we actually don't add infinitely many numbers.

Remark: Series addition is NOT commutative.

☐ **Theorem 2.3**: A series converges **absolutely** if the series of absolute values converges. If so it converges.

Proof: Eventually the partial sums of the absolute values will differ by $|a_i| + |a_i + 1| + \ldots |a_j| < M$ which is bounded. Then $M < a_i + \ldots a_j < M$ so the series is Cauchy and thus converges in $\mathbb{R}$. ∎

☐ **Theorem 2.4**: Rearranging the terms of a absolutely convergent series will not change its sum.

Proof: Quite fun to prove so I'll make you do it yourself again. ∎

☐ **Theorem 2.5**: For a series that does not converge absolutely, (**conditionally**) any limit may be obtained by rearranging the terms.

Limits of real functions

$$\lim_{x \to p} f(x) = L$$

if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - p| < \delta$.

☐ **Theorem 2.6**: f is continuous at x if the limit at x exists and is equal to $f(x)$.

The limit as $x \to \infty$ is defined similarly .

3 Differentiation

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Remark: I prefer $\frac{f(y)-f(x)}{y-x}$ it's symmetric and a bit easier to write.

This limit only exists if the function is continuous at x.

Local maximum A continuous $f : U \to R$ has a local maximum at $x \in U$ if $f(x) \geq f(y)$ for all y in some open interval containing x.

A local minimum is defined similarly.

☐ **Theorem 3.1** (Fermat's Theorem): If f is differentiable everywhere then for every local extrema x, $f'(x) = 0$.

Proof: Consider a local maxima x in any open neighbourhood of x, $f(y) > f(x)$, we can take both $y > x$ and $y < x$ in this neighbourhood. In either case $\frac{f(y)-f(x)}{y-x}$ $<$ and > 0. So the limit must be 0, as otherwise if it's t we can pick an epsilon $< t$ and get a contradiction! ∎

This is why we shouldn't consider boundary points to be local extrema, because this proof would not work.

☐ **Theorem 3.2** (Rolle's Theorem): If $f : [a, b] \to \mathbb{R}$ is a continuous function which is differentiable on the open interval (a, b) and $f(a) = f(b)$, then there exists some $c \in (a, b)$ where $f'(c) = 0$.

Proof: $[a, b]$ is compact therefore $f[a, b]$ is compact too hence bounded from above and below and so there exists a global maximum and minimum. Either a, b are both maximums and minimums or there is a global maximum / minimum somewhere in (a, b) and hence a local maximum / minimum. We are done by ☐ Theorem 3.1. ∎

☐ **Theorem 3.3** (Mean value Theorem): If $f : [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b) then there is a point c where $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof: Define $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}x$. Then apply ☐ Theorem 3.2 to g. ∎

☐ **Corollary 3.3.1** (Racetrack Principle): If $f'(x) \leq g'(x)$ for every $x > 0$, then $f(x) \geq g(x)$ for every $x > 0$.

Proof: Take the function $h(x) = f(x) - g(x)$. Then $h'(x) \leq 0$ so $h(0)$ is 0. If there is any point where $h(x) < 0$ then the secant line $0 \longleftrightarrow x$ has negative gradient but $h(x)$ has positive gradient at some point by the mean value theorem. Contradiction!. ∎

Smoothness A function is smooth if it has derivatives of all orders.

Example (Intuition Check): The following function has all derivatives at $0 = 0$.

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x > 0 \end{cases}$$

☐ **Theorem 3.4** (Jensen's Inequality): $f : (a, b) \to \mathbb{R}$ is twice differentiable and $f''(x) \geq 0$ for all x. $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$.

Proof: $(f')'(x) \leq 0$ the function $f'(x)$ must be decreasing as otherwise by □ Theorem 3.3 a secant line of $f'(x)$ would have positive gradient and hence $f''(x)$ is positive at some point. Contradiction!

Now assume if $f\left(\frac{x+y}{2}\right) > \frac{f(x)+f(y)}{2}$ then $f(y) - f\left(\frac{x+y}{2}\right) > f\left(\frac{x+y}{2}\right) - f(x)$. The secant line $x \longleftrightarrow \frac{x+y}{2}$ has smaller gradient than the secant line $\frac{x+y}{2} \longleftrightarrow y$. Then by □ Theorem 3.3 there is some point in $\left(x, \frac{x+y}{2}\right)$ where it's gradient is less than some point in $\left(\frac{x+y}{2}, y\right)$. Contradiction!. ∎

4 Power series and Taylor series

A power series is the infinite sum

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

Radius of Convergence

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$$

with the convention that $R = 0$ if the left hand side converges.

Analytic A function is analytic at x if it is equal to a power series in some neighbourhood of x.

For a smooth function the series $\sum \frac{f^{(n)}(p)}{n!} z^n$ is it's Taylor Series. If it's analytic the power series is exactly this Taylor Series.

Example (Resolution): The following function has all derivatives at $0 = 0$, but it's not analytic.

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x > 0 \end{cases}$$

5 Riemann Integration

Uniformly Continuous A continuous function over metric spaces is uniformly continuous if for all ε there is δ so that

$$d_M(p, q) < \delta \Rightarrow d_N(f(p), f(q)) < \varepsilon$$

the difference is that δ does not depend on p or q.

☐ **Theorem 5.1** (Compact Spaces imply Uniform Continuity): A continuous function $f : M \longrightarrow N$ where M is a compact metric space is uniformly continuous.

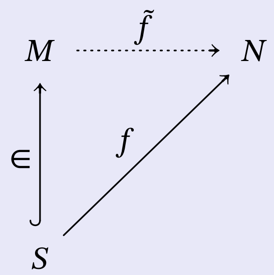
Proof: Suppose for some value of ε we can never find a corresponding value of δ, hence for all δ there exists pairs (p, q) such that $d_{M(p,q)} < \delta$ but $d_{N(f(p),f(q))} \geq \varepsilon$

Consider when delta is $1, \frac{1}{2}, \frac{1}{3}, \ldots$, this gives us the pair of sequences (p_n, q_n) such that $d_{M(p_n,q_n)} < \frac{1}{n}$ but $d_N(f(p_n), f(q_n)) \geq \varepsilon$. A subsequence of p_n converges to some x and so the corresponding subsequence of q_n must also converge to x. Which means the corresponding f sequences converge to the same points too, but they don't because they differ by epsilon. ∎

Dense A subset X of a topological space is dense if every open subset of S contains a point of X.

Example: The rationals are dense in the reals.

☐ **Theorem 5.2** (Extending uniformly continuous functions): Let M be a metric space and S a dense subset of M. If $f : S \longrightarrow N$ is uniformly continuous function.



Then there is exists a unique continuous function $\tilde{f}$ such that the diagram commutes.

Proof: Construct a cauchy sequence in S which converges to $x \in M$ by picking smaller and smaller ε neighbourhoods around x.

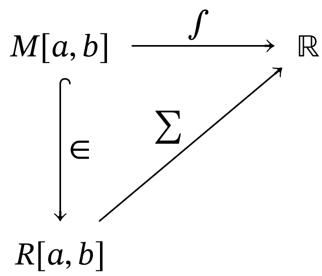
It is possible to prove that the f of this sequence converge to something in N due to uniform continuity. Define $\tilde{f}(x)$ to be this limit. ∎

Remark: This is such a cool way to set up integration by the way.

Let $M[a,b]$ be the set of continuous functions over the interval $[a,b]$ as well as the set of rectangle functions. We'll denote the rectangle functions as $R[x,y]$.

This is a metric space with the metric $\sup|f(x) - g(x)|$. The set of rectangle functions is dense over $M[a,b]$.

The integral of a rectangle function is the sum of the areas of the rectangles. We will let this be $\sum : R[x,y] \to \mathbb{R}$.



This is how the definite integral is defined. In words it is the limit of the area under the rectangle functions which approximate the function.